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D-instanton induced interactions on a D3-brane

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Abstract

Non-perturbative features of the derivative expansion of the effective action of a single D3-brane are obtained by considering scattering amplitudes of open and closed strings. This motivates expressions for the coupling constant dependence of world-volume interactions of the form $(\partial F)^4$ (where F is the Born–Infeld field strength), $(\partial^2\varphi)^4$ (where φ are the normal coordinates of the D3-brane) and other interactions related by $\mathcal{N} = 4$ supersymmetry. These include terms that transform with non-trivial modular weight under Montonen–Olive duality. The leading D-instanton contributions that enter into these effective interactions are also shown to follow from an explicit stringy construction of the moduli space action for the D-instanton/D3-brane system in the presence of D3-brane open-string sources (but in the absence of a background antisymmetric tensor potential). Extending this action to include closed-string sources leads to a unified description of non-perturbative terms in the effective action of the form (embedding curvature)^2 together with open-string interactions that describe contributions of the second fundamental form.

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1. Introduction

This paper concerns properties of scattering amplitudes of open and closed strings on a D-brane and their implications for the low energy effective world-volume action. We will be particularly interested in non-perturbative effects associated with the presence of D-instantons (or D(-1)-branes) which are essential in ensuring the $SL(2,Z)$ invariance of type IIB string theory. The scattering of open string states describes the interactions of both the Born–Infeld world-volume gauge field and of the scalar world-volume fields, while interactions between closed and open strings describe gravitational effects induced on the brane [1,2]. These curvature-dependent effects result both from the embedding of the D-brane in a geometrically non-trivial target space as well as from the non-trivial intrinsic geometry of the D-brane. The long wavelength dependence of such effects is encoded in the derivative expansion of the effective world-volume action of the D-brane. Of course, the Dirac–Born–Infeld (DBI) part of the D-brane action already contains an infinite number of higher derivative terms since it can be expanded in an infinite power series in the Born–Infeld field strength, F. However, this only accounts for constant F's while we will be concerned with terms that depend on the first derivative of F [3,4]. These arise as natural partners of terms in the world-volume action such as those of the form R^2, which denotes the sum of a number of (curvature)^2 terms [5]. These include both normal and tangential components of the pull-back of the curvature together with the contributions that come from the second fundamental form (which depends on the scalar world-volume fields) that enter for non-geodesic embeddings.

We will focus particularly on properties of the D3-brane, which is a system of obvious intrinsic interest in the context of four-dimensional field theory. The requirement that the equations of motion derived from the action be invariant under $SL(2,Z)$ duality transformations provides very strong constraints on the possible structure of higher dimensional terms in the world-volume action, just as in the case of the bulk effective string action [6,7]. As we will see, this motivates a non-perturbative expression for low-lying terms in the derivative expansion of the action that includes an exact description of the effects of D-instantons that are localized on the D3-brane.

In section 2 we will review the amplitude that describes the scattering of four massless open-string states on a D3-brane. The low energy expansion of the well-known expression for the tree amplitude leads to contributions in the effective action that are of order α'^4 relative to the classical Yang–Mills amplitude.[1] These terms include one of the form $(\partial F)^4$ and one of the form $(\partial^2\varphi)^4$, where the six scalar fields φ^a $(a = 1, \ldots, 6)$ describe the transverse oscillations of the D3-brane (and the contractions between the fields and the derivatives will be specified later). It is easy to argue that these interactions also receive corrections at one string loop [8,9] but they are not expected to get corrections from

[1] We will always count powers of α' relative to the F^2 term in this paper.

higher-order perturbative effects. The invariance of the effective action under $SL(2, Z)$ Montonen–Olive duality transformations of the complex coupling constant will also be discussed in section 2. This requires that the dependence on the complex coupling constant (the type IIB complex scalar field, τ) enters the interaction via a modular invariant prefactor, $h(\tau, \bar{\tau})$. We will argue that the known perturbative contributions to the higher derivative interactions are consistent with $h(\tau, \bar{\tau})$ having the form,

$$h(\tau, \bar{\tau}) = \ln |\tau_2 \eta(\tau)^4|, \tag{1.1}$$

where η is the Dedekind function and τ is the complex background scalar field $\tau = \tau_1 + i\tau_2 = C^{(0)} + ie^{-\phi}$. The string coupling is $g = e^{\phi}$ where ϕ is the type IIB dilaton and $C^{(0)}$ is the Ramond–Ramond ($R \otimes R$) scalar. The function h has the weak coupling (large τ_2) expansion,

$$h(\tau, \bar{\tau}) = \Big(-\frac{\pi}{3}\tau_2 + \ln \tau_2 - 2 \sum_{N=1}^{\infty} \sum_{m|N} \frac{1}{m} \big(e^{2\pi i \ N\tau} + e^{-2\pi i \ N\bar{\tau}} \big) \Big), \tag{1.2}$$

which contains the expected power-behaved terms that are identified with tree-level and one-loop terms of open-string perturbation theory, as well as a specific infinite set of D-instanton corrections which will be discussed in the following sections. The function (1.1) is the same modular function that appears in several other contexts [5,7,10,11,12,13,14,15].

In section 3 we will describe the bosonic and fermionic collective coordinates of a single D3-brane in the presence of a D-instanton. This is a 1/4-BPS system that preserves eight of the 32 components of the ten-dimensional type II supersymmetry. As in [16,17,18] we will motivate the description of the collective coordinates by considering the toroidal compactification of the composite Dp/D$(p+4)$ system. We will make particular reference to the D5/D9 system as a simple way of enumerating the open-string fields and their interactions. Upon toroidal compactification combined with T-duality this reduces to other well-studied systems such as the D0/D4 system in which the D0-brane is described by supersymmetric quantum mechanics on instanton moduli space [17,19]. Another well-studied example is the D1/D5 system in which the string is described by a two-dimensional $(4, 4)$ supersymmetric sigma model with the instanton moduli space as the target manifold [20]. Compactification and T-duality on T^6 leads to a description of the D-instanton/D3-brane system in which the open strings that end on the D-instanton describe the isolated states corresponding to collective coordinates rather than to fields. It is essential to integrate over these coordinates in evaluating scattering amplitudes. We will emphasize the origin of these collective coordinates and their interactions from the insertion of open string vertex operators on a world-sheet that is a disk with a segment of 'Neumann' boundary (on which the D3-brane boundary conditions are satisfied) and a segment of 'Dirichlet' boundary (on which the D-instanton boundary conditions are satisfied). This gives an

efficient procedure for describing the eight components of unbroken supersymmetry and the twenty-four components of broken supersymmetry of the system.

In section 4 we will consider the scattering of open-string states on the D3-brane in the background of a D-instanton to lowest order in the string coupling. This involves the coupling of open-string sources to the Neumann boundary segments, from which we deduce the moduli space action for the D-instanton in the presence of a D3-brane source. This action is explicitly invariant under the twenty-four non-linearly realized broken supersymmetries as well as eight linearly realized unbroken supersymmetries. The unbroken supersymmetry transformations of the three-brane fields (the $\mathcal{N}=4$ Maxwell multiplet) differ from those of the free field theory by a simple term involving the collective coordinates of the D(-1)/D3 system.

This action defines a generating function for the leading perturbative contribution to the correlation functions of massless open-string states in a D-instanton background, which will be considered in section 5. Integration over the collective coordinates gives the leading order (in powers of the string coupling) D-instanton contribution to scattering amplitudes of ground-state open strings on the D3-brane. These determine terms in the effective world-volume action that are proportional to the higher-derivative interactions, such as the $(\partial F)^4$ and $(\partial^2\varphi)^4$, that arose in the tree-level analysis of section 2. These terms have a dependence on the coupling of the form $e^{2\pi i\tau}$, which agrees with the leading $N=1$ instanton term in the expansion (1.2) of the conjectured exact form of the modular invariant effective action. This analysis extends to the $SL(2,Z)$-invariant interactions of the form $(\partial F)^2(\partial^2\varphi)^2$, $F^+\partial^2F^-\partial^2\Lambda\partial\bar{\Lambda}$, $(\partial\bar{\Lambda})^2(\partial^2\Lambda)^2$ and others, where Λ^A_α and $\bar{\Lambda}^{\dot{\alpha}}_A$ are the Weyl fermions of the $\mathcal{N}=4$ theory.

More generally, as will be evident in section 6, there are interactions at the same order in α' that transform under $SL(2,Z)$ with non-zero modular weight. Examples of these are interactions of the form $(\partial F)^2(\partial\bar{\Lambda})^4$ (which transforms with holomorphic and anti-holomorphic weights $(-1,1)$) and $(\partial\bar{\Lambda})^8$ (which transforms with holomorphic and anti-holomorphic weights $(-2,2)$). Correspondingly, the coupling constant must enter in prefactors, $h^{(1,-1)}(\tau,\bar{\tau})$ and $h^{(2,-2)}(\tau,\bar{\tau})$, that are modular forms of compensating weights. We will argue that supersymmetry together with $SL(2,Z)$ invariance requires that these prefactors are given by applying appropriate modular covariant derivatives to the modular function h. However, this remains a conjecture that should be justified by a deeper understanding of the constraints imposed by $\mathcal{N}=4$ supersymmetry.

The open-string calculations of the earlier sections combine naturally with closed-string D-instanton effects that describe the coupling of bulk gravity to the world-volume of the D3-brane. There are two effects of this kind. One of these, to be described in detail in section 7, arises from the coupling of closed strings to the D-instanton. Just as in the bulk theory [6] the leading perturbative contribution of this kind is associated with the coupling of a closed-string vertex operator to disk diagrams with purely Dirichlet boundary

conditions and with fermionic open strings attached. These fermionic strings describe the sixteen fermionic moduli of the bulk D-instanton (so that, for example, an R^4 term is generated in the bulk theory from the product of four such disks, each with a graviton and four fermionic open strings attached). Integration over the collective coordinates of the D(-1)/D3 system soaks up eight of these fermionic moduli. The remaining eight fermionic moduli generate D-instanton contributions to R^2 terms which will be evaluated explicitly. These have previously been obtained by indirect arguments [5]. We will see that these instanton contributions package together with the open-string interactions to give the nonperturbative generalization of the complete tree-level R^2 term. This term includes the effects of nongeodesic embeddings of the D3-brane in a general target space which involves contributions of the second fundamental form.

The second effect arises from the coupling of the $NS \otimes NS$ antisymmetric tensor potential (B) to the combined D(-1)/D3 system and is described by attaching a closed-string B vertex operator to the disk with two boundary twist operators. This coupling combines with the analogous coupling of the Born–Infeld field in the usual gauge-invariant combination, $B + 2\pi\alpha' F$. With a non-vanishing background value for B the instanton give a non-trivial $\alpha' \to 0$ decoupling limit, even in the abelian case. It is well known [21] that for $B \neq 0$ the singular abelian instanton of $\mathcal{N} = 4$ Maxwell theory is regularized and is equivalent to a Fayet-Illipoulos deformation of the ADHM moduli space that removes the small instanton singularity [22]. Such effects, which will not be discussed very much in this paper, should generate interactions with fewer derivatives than those considered here.

2. The low energy dynamics of D3-brane excitations

At long wavelengths the dynamics of the excitations on a Dp-brane may be well approximated by an action that consists of the sum of the Dirac–Born–Infeld (DBI) action and a Wess–Zumino (WZ) term,

$$S_p = S_p^{DBI} + S_p^{WZ}, \tag{2.1}$$

The DBI term takes the form

$$S_p^{DBI} = T_{(p)} \int d^{p+1}x \, \sqrt{\det\big((G_{\mu\nu} + B_{\mu\nu})\partial_m Y^\mu \partial_n Y^\nu + 2\pi\alpha' \, F_{mn}\big)}, \tag{2.2}$$

where F_{mn} is the world-volume Born–Infeld field strength, $G_{\mu\nu}$ is the target-space metric, the tension $T_{(p)}$ is given by

$$T_{(p)} = 2\pi(4\pi^2\alpha')^{-(1+p)/2} \, e^{-\phi}, \tag{2.3}$$

the antisymmetric tensor potential, $B_{\mu\nu}$, will be set equal to zero in the remainder of this paper. With this definition the Yang-Mills coupling constant has the value

$$g_{p+1}^2 = 4\pi(4\pi^2\alpha')^{(p-3)/2}\, e^{\phi}, \tag{2.4}$$

as can be seen from the expansion of (2.2) to order F^2. Our conventions are that ten-dimensional space-time vectors are labeled by $\mu = 0, \ldots, 9$ while a $SO(9,1)$ Majorana–Weyl spinor will be labeled $\mathcal{A} = 1, \ldots 16$. The world-volume directions are $m, n, \ldots = \mu = 0, \ldots, p$, while the directions transverse to the Dp-brane will be labeled $a, b, \ldots = p+1, \cdots, 9$. The low energy limit of the WZ term in the action (2.1) can be deduced by requiring the absence of chiral anomalies in an arbitrary configuration of intersecting D-branes [5,23,24,25]. In static gauge the tangential components of the embedding coordinates are identified with the world-volume coordinates, while the normal components are identified with the scalar world-volume fields.

$$Y^m(x^m) = x^m, \qquad Y^a(x^m) = 2\pi\alpha'\ \varphi^a(x^m) \tag{2.5}$$

Where φ^a is a canonically normalized scalar field of dimension $[L]^{-1}$, (whereas the coordinate Y has dimension $[L]$). More generally, one defines normal and tangent frames as summarized in appendix B.

The DBI action is exact only in the approximation that all derivatives of the field strength ∂F and second derivative (acceleration) terms on scalars $\partial^2\varphi$ are small enough to be ignored [26]. Terms of higher order in derivatives may be studied by considering the scattering of open strings (or ripples) propagating on the D-brane. The tree-level amplitude for the scattering of four ground-state open superstrings on a Dp-brane with any permitted value of p is given by compactification of the familiar expression,

$$A_4^{tree} = \mathcal{N} K \left\{ \frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)}{\Gamma(1-\alpha' s - \alpha' t)} + (s \leftrightarrow u) + (t \leftrightarrow u) \right\}, \tag{2.6}$$

where the kinematic factor K is of the form

$$K = -16 t_8^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} \zeta^{(1)}_{\mu_1} k^{(1)}_{\nu_1} \zeta^{(2)}_{\mu_2} k^{(2)}_{\nu_2} \zeta^{(3)}_{\mu_3} k^{(3)}_{\nu_3} \zeta^{(4)}_{\mu_4} k^{(4)}_{\nu_4}, \tag{2.7}$$

where $\mu, \nu = 0, \cdots, 9$ and t_8 is a well-known eighth-rank tensor that is an appropriately symmetrized sum of products of four Kronecker δ's [27]. The normalization factor is given by,

$$\mathcal{N} = -\frac{1}{16\pi} {\alpha'}^2 e^{-2\phi}. \tag{2.8}$$

In order to apply this formula to the scattering of open-string excitations of a three-brane in static gauge, one simply restricts the momenta $k^{(r)}_{\mu_r}$ to lie in the world-volume directions for which $\mu_r = a_r = 0, 1, 2, 3$. The polarization vectors of the world-volume

vector field are denoted $\zeta^{(r)}_{a_r}$, $a_r = 0,1,2,3$ whereas the six scalars φ have wave functions $\zeta^{(r)}_a$, $a = \mu - 3 = 1, \cdots, 6$. For the transverse scalars the kinematic factor is particularly simple

$$K = -16\Big(st\zeta_1\cdot\zeta_3\ \zeta_2\cdot\zeta_4 + tu\zeta_1\cdot\zeta_2\ \zeta_3\cdot\zeta_4 + su\zeta_2\cdot\zeta_3\ \zeta_1\cdot\zeta_4\Big). \tag{2.9}$$

Using the expansion of the logarithm of the Gamma function

$$\ln\Gamma(1+z) = -Cz + \sum_{k=2}^{\infty}(-1)^k\frac{z^k}{k}\zeta(k), \tag{2.10}$$

and using $s+t+u=0$ it is easy to see that the ratio of Γ's in (2.6) has the expansion,

$$\begin{aligned}\frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)}{\Gamma(1-\alpha' s-\alpha' t)} &= \frac{4}{\alpha'^2 st}\exp\Big(-\frac{\zeta(2)}{4}\alpha'^2 st - \frac{\zeta(4)}{32}\alpha'^4 st(2t^2+2s^2+3st) + o(s^4)\Big) \\ &= \frac{4}{\alpha'^2 st} - \zeta(2) + \frac{\zeta(2)^2}{8}\alpha'^2 st - \frac{\zeta(4)}{8}\alpha'^2(2t^2+2s^2+3st) + o(s^4)\end{aligned}. \tag{2.11}$$

The first term on the right-hand side describes massless tree level exchange that arises from the Yang–Mills action that is obtained as the leading term in the expansion of S_3 in powers F. However, these pole terms cancel in the abelian case of relevance to us after the three terms in (2.6) are added, since the scattering states carry no charge. The second term on the right-hand side of (2.11) corresponds to the F^4 term in the expansion of the DBI action while the third and fourth terms correspond to higher derivative interactions of the form $(\partial F)^4$ and $(\partial^2\varphi)^4$. The four-point amplitude has a series expansion in powers of spatial derivatives of the form

$$A_4^{tree} = A_4^{(0)} + \alpha'^2 A_4^{(2)} + \alpha'^4 A_4^{(4)} + \dots. \tag{2.12}$$

After adding the three different orderings, the $A_4^{(4)}$ terms in the amplitude for scattering four transverse scalars are given by

$$\begin{aligned}A_4^{(4)} &= \frac{\mathcal{N}K}{\alpha'^2}\left(\frac{\zeta(2)^2}{8} + \frac{5\zeta(4)}{8}\right)(st+tu+su) \\ &= -\frac{\pi^3 e^{-\phi}}{3\times 2^6}\left(s^2+t^2+u^2\right)\left(st\zeta_1\cdot\zeta_3\ \zeta_2\cdot\zeta_4 + tu\zeta_1\cdot\zeta_2\ \zeta_3\cdot\zeta_4 + su\zeta_2\cdot\zeta_3\ \zeta_1\cdot\zeta_4\right).\end{aligned} \tag{2.13}$$

2.1. Higher derivative terms and $SL(2,Z)$ invariance

A consequence of S-duality of IIB superstring theory is that the D3-brane is inert under $SL(2,Z)$ transformations which act on τ by

$$\tau \to \frac{a\tau+b}{c\tau+d}, \tag{2.14}$$

where $a, b, c, d \in Z$ and $ad - bc = 0$. The metric tensor in the Einstein frame is inert under this transformation while the antisymmetric two-form potentials transform as a $SL(2, Z)$ doublet.

Such an S-duality transformation is a symmetry of the equations of motion for the D3-brane that come from the variation of the sum of the D3-brane DBI and WZ actions, when it is accompanied by a $SL(2, Z)$ electromagnetic duality transformation of the world-volume fields [28,29]. The field strength, F, together with its dual, $G^{mn} = i\delta S_3/\delta F_{mn}$, form an $SL(2, Z)$ doublet which transforms as,

$$\begin{pmatrix} *G \\ F \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} *G \\ F \end{pmatrix}. \tag{2.15}$$

The combinations

$$F^+ = \frac{1}{i\tau_2}(\tau F - *G), \qquad F^- = \frac{1}{i\tau_2}(\bar{\tau} F - *G) \tag{2.16}$$

transform as forms of weight $(1, 0)$ (for F^-) and $(0, 1)$ (for F^+)[2] so that

$$F^- \to (c\tau + d)\, F^-, \qquad F^+ \to (c\bar{\tau} + d)\, F^+, \tag{2.17}$$

which means that the combination

$$\tau_2 F^+ F^- \tag{2.18}$$

is invariant under $SL(2, Z)$. At lowest order in the low energy expansion G^{mn} is given by its Maxwell form, $G_{mn} = ie^{-\phi} F_{mn} + C^{(0)} \frac{1}{2} \epsilon_{mnpq} F^{pq}$ and the expressions $F^{\pm}$ are the self-dual and anti self-dual field strengths,

$$F^{\pm} = F \pm *F. \tag{2.19}$$

These formulae are appropriate for euclidean signature for which $** = 1$.

The preceding discussion of modular invariance only applies in the approximation that the Born–Infeld field F is constant so that the action S_3 is valid. However, the issue of $SL(2, Z)$ invariance of the D3-brane when F is not constant, so that the derivative of F is non-zero, requires further investigation. The low-energy expansion of the four-point function (2.13) determines such higher derivative corrections. The first terms (with bosonic open-string fields) that arise beyond the F^4 terms have the schematic form,

$$S_3' = \frac{\pi^3 \alpha'^4}{12} \int d^4x \sqrt{g}\, \tau_2 \Big((\partial^2 \varphi)^4 + (\partial^2 \varphi)^2\, \partial F^+ \partial F^- + (\partial F^+)^2 (\partial F^-)^2 \Big), \tag{2.20}$$

[2] In this notation a modular form of weight (p, q) transforms with holomorphic weight p and anti-holomorphic weight q.

where the exact tensor structure of contractions of each term is determined by demanding that (2.20) reproduces (2.13) when transformed into momentum space. The canonically normalized scalar field φ is related to the embedding coordinate Y by (2.5).

In order to exhibit the transformation properties of these higher derivative terms under $SL(2,Z)$ it is convenient to transform the world-volume metric to the Einstein frame using $g_{\mu\nu} = \tau_2^{-1/2} g^{(E)}_{\mu\nu}$, where $g^{(E)}_{\mu\nu}$ is the modular invariant Einstein frame metric. In the Einstein frame (2.20) can be expressed as

$$S'_3 = \frac{\pi^3 \alpha'^4}{12} \int d^4x \sqrt{g^{(E)}}\, \tau_2 \Big((\partial^2\varphi)^4 + \tau_2 (\partial^2\varphi)^2\, \partial F^+ \partial F^- + \tau_2^2 (\partial F^+)^2 (\partial F^-)^2 \Big). \quad (2.21)$$

In order to avoid complications we will specialize to the case in which τ is constant. The equations of motion that follow from the total action $S_{BI} + S_{WZ} + S'_3$ fail to be $SL(2,Z)$ invariant only because of the overall factor τ_2 in S'_3. To see this it is important to note that the higher derivative terms in (2.21) modify the expression for G and hence the transformation rules (2.15) to the fixed order of α' we are considering. This means that in order for the full nonperturbative expression to be modular invariant this overall factor of τ_2 must be replaced by a modular function, $h(\tau, \bar\tau)$, leading to

$$S'_3 = \frac{\pi^2 \alpha'^4}{4} \int d^4x \sqrt{g^{(E)}} h(\tau,\bar\tau) \Big((\partial^2\varphi)^4 + \tau_2 (\partial^2\varphi)^2\, \partial F^+ \partial F^- + \tau_2^2 (\partial F^+)^2 (\partial F^-)^2 \Big). \quad (2.22)$$

Apart from the tree-level contribution to these four-string processes there is also a logarithmically divergent one-loop contribution. This arises, for example, in a field theory calculation of the $\langle F^2(x_1) F^2(x_2) \rangle$ correlation function, in which the one-loop diagram has two F^4 vertices [8,9]. Similarly, it is straightforward to see that there is a logarithmic infrared divergence in the open string one-loop amplitude. These observations make it plausible that the function h is proportional to (1.1) which has the the weak coupling expansion (1.2). Support for this ansatz is reinforced by the fact that the interactions in (2.22) combine naturally with the induced (curvature)^2 terms discussed in [5] for which the prefactor h was proportional to (1.1). The presence of terms in (2.22) of the form $(\partial^2\varphi)^4$ in the D3-brane action have a natural geometric origin in terms of the contribution of the second fundamental form to the action for a D3-brane embedded in a curved target space, as will be seen in section 7.

3. The D3-brane in the presence of a D-instanton

3.1. Collective coordinates

The conventional instanton moduli space arises in the Higgs branch of the gauge theory and has a natural D-brane interpretation [20]. In the following we will be describing

the case of a single D3-brane so the world-volume gauge theory is abelian and there is no Higgs branch and no 'fat' instantons. However, within string theory there is a well-defined prescription for describing the moduli space of such a 'pointlike' instanton in terms of the configuration of open strings joining the D3-brane to a D-instanton at a fixed transverse separation [30,31,32,33].

Following [18,20,23] we may consider the D-instanton/D3-brane system to be obtained by T-duality from the D5/D9 system compactified on a six-torus. The $SO(10)$ of the euclidean ten-dimensional theory is broken in this background to $SO(6) \times SO(4) \sim SU(4) \times SU(2)_L \times SU(2)_R$, where the $SO(6) \sim SU(4)$ is the rotation group in the D5-brane which is the R-symmetry group of $\mathcal{N} = 4$ Yang–Mills theory, while the $SO(4) \sim SU(2)_L \times SU(2)_R$ is the rotation group in the directions transverse to the D5-brane.[3] Making six T-dualities in the directions $\mu = 4, \cdots, 9$ transforms the D9-brane into a D3-brane filling the $\mu = n = 0, 1, 2, 3$ directions. After turning on Wilson lines the D-instanton is located at a space-time point separated from the D3-brane in the directions $a = \mu - 3 = 1, \ldots, 6$. We will label the ADHM supermoduli using standard notation (as, for example, in [18,34]).

The dynamics of a single Dp-brane is determined by the reduction to $p+1$ space-time dimensions of the ten-dimensional supermultiplet of Maxwell theory, $(A_\mu, \Psi^{\mathcal{A}})$, where $\mathcal{A}$ denotes the sixteen components of a Majorana–Weyl spinor.[4] Thus, the massless world-volume fields of the D3-brane and the collective coordinates of the D-instanton are given by dimensional reduction of the ten-dimensional theory to four and zero dimensions respectively. In the case of the D3-brane the bosonic fields comprise the six scalars and a gauge field

$$A_\mu = (\varphi_{AB}, A_n). \tag{3.1}$$

Here A_n transforms as $(\mathbf{1}, \mathbf{2}, \mathbf{2})$ of $SU(4) \times SU(2)_L \times SU(2)_R$ while φ_{AB} transforms as $(\mathbf{6}, \mathbf{1}, \mathbf{1})$, where the six transverse scalars $\varphi^a, a = 1, \cdots, 6$ are related to φ_{AB} by

$$\varphi_{AB} = \frac{1}{\sqrt{8}} \Sigma^a_{AB} \varphi^a = \frac{1}{2} \epsilon_{ABCD} \bar{\varphi}^{CD}. \tag{3.2}$$

(where the Clebsch–Gordon coefficients Σ^a_{AB} are defined in appendix A). The massless fermionic D3-brane fields are given by the spinors

$$\Psi = (\Lambda^A_\alpha, \bar{\Lambda}^{\dot{\alpha}}_A), \tag{3.3}$$

[3] The defining representations of $SU(4) \times SU(2)_L \times SU(2)_R$ will be labeled by the indices $A = 1, 2, 3, 4$, $\alpha, \dot{\alpha} = 1, 2$.

[4] In the following the euclidean continuation of a ten-dimensional Majorana fermion is chosen to be real. Since the conjugate spinor is not real, this leads to a non-hermitian euclidean hamiltonian, which is not a problem since hermiticity is not relevant in euclidean fields theory.

where the gauginos $\Lambda^A_\alpha, \bar\Lambda^{\dot\alpha}_A$, transform as $(\mathbf{4},\mathbf{2},\mathbf{1})$ and $(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})$, respectively.

Similarly the collective coordinates of an isolated D-instanton are determined by the reduction of super Yang–Mills to zero space-time dimensions. The bosonic vector potential decomposes under $SO(4)\times SO(6)$ as

$$A_\mu = (a_n, \chi_a), \tag{3.4}$$

($n = 0,1,2,3$ and $a = 1,\cdots,6$). In terms of the covering group the bosonic fields are χ^a, which forms a $(\mathbf{6},\mathbf{1},\mathbf{1})$, and $a_{\alpha\dot\beta} = \sigma^n_{\alpha\dot\beta} a_n$ which forms a $(\mathbf{1},\mathbf{2},\mathbf{2})$. The vector χ^a is also conveniently written as a 4×4 antisymmetric matrix, χ_{AB}, which is defined by

$$\chi_{AB} = \frac{1}{\sqrt{8}}\Sigma^a_{AB}\chi_a. \tag{3.5}$$

The D-instanton fermions arise from the ten-dimensional Majorana–Weyl spinor which decomposes as $\mathbf{16}\to(\mathbf{4},\mathbf{2},\mathbf{1})\oplus(\bar{\mathbf{4}},\mathbf{2},\mathbf{1})$ so that

$$\Psi = (M'^A_\alpha, \lambda_{A\dot\alpha}). \tag{3.6}$$

Although the action for a single D5-brane is trivial it becomes non-trivial in the presence of the D9-brane, which breaks the six-dimensional $(1,1)$ supersymmetry to $(0,1)$. The interactions arise due to the presence of massless fields associated with the open strings joining the D5-brane and the D9-brane. These are the $(0,1)$ hypermultiplets $(\mu^A, w_{\dot\alpha})$ and $(\bar\mu^A, \bar w^{\dot\alpha})$ which consist of the bosonic fields $w_{\dot\alpha}$ and $\bar w^{\dot\alpha} = (w_{\dot\alpha})^*$ in $(\mathbf{1},\mathbf{1},\mathbf{2})$ and their fermionic partners μ^A and $\bar\mu^A$ in $(\mathbf{4},\mathbf{1},\mathbf{1})$. After T-duality on the six-torus the separation of the D3-brane at the origin and the D(-1)-brane is given by $\chi^2 \equiv \chi^a\chi_a = L^2$, which is the length of the stretched strings joining the two D-branes.

In addition to these fields it is usual to introduce the anti-self-dual auxiliary field, $D_{mn} = -(*D_{mn})$, in order to make supersymmetry manifest. The self-duality property means that this field transforms in the adjoint of $SU(2)_R$ and can be rewritten in terms of D^c ($c = 1,2,3$) using $D_{mn} = -D^c\bar\eta^c_{mn}$, where η^c_{mn} is the 'tHooft symbol defined in appendix A.

The form of the action for the D-instanton collective coordinates follows from the toroidal compactification of the $d=6$ $N=1$ world-volume theory of a single D5-brane in the presence of a D9-brane and has the form (see, for example, [18] with $k=N=1$),

$$S_{-1} = -2\pi i\tau + \frac{1}{g_0^2}D_c^2 + iD^cW^c + \chi^2 W_0 - 2i\bar\mu^A\mu^B\chi_{AB} + i\left(\bar\mu^A w_{\dot\alpha}\lambda^{\dot\alpha}_A + \bar w_{\dot\alpha}\mu^A\lambda^{\dot\alpha}_A\right), \tag{3.7}$$

where

$$W_0 = \bar w w, \qquad W^c = \bar w\bar\sigma^{mn}w\,\bar\eta^c_{mn} \equiv W^{mn}\,\bar\eta^c_{mn}, \tag{3.8}$$

(where $W^{mn} = \bar{w}\bar{\sigma}^{mn}w$) which satisfy the constraint

$$W_0^2 = (W^c)^2 = W_{mn}W^{mn}. \tag{3.9}$$

The quantity g_0 in (3.7) is the zero-dimensional coupling constant, which is given from (2.4) by

$$g_0^2 = 4\pi(4\pi^2\alpha')^{-2}\, e^{\phi}, \tag{3.10}$$

and τ is the constant value of the complex bulk scalar field. It is notable that there are no couplings of the a_n and ${M'}^A_\alpha$ in the action (3.7), which arises from the fact that these are the supermoduli associated with the relative longitudinal position of the D-instanton and the D3-brane.

The auxiliary field D^c may be integrated out, producing a factor of g_0^3 in the measure which will be important later. The resulting moduli-space action of the abelian D-instanton becomes

$$S_{-1} = -2\pi i\tau + \frac{g_0^2}{4}(W^c)^2 + \chi^2 W_0 - 2i\bar{\mu}^A\mu^B\chi_{AB} + i\left(\bar{\mu}^A\, w\lambda_A + \mu^A\, \bar{w}\lambda_A\right). \tag{3.11}$$

In order to recover the usual limit of $\mathcal{N} = 4$ superconformal Yang–Mills theory in four dimensions it is necessary to decouple the closed string sector [35] by taking the low energy limit $\alpha' \to 0$. Since $g_0^2 \sim e^{\phi}/\alpha'$ this means taking $g_0^2 \to \infty$ with fixed string coupling e^{ϕ} which enforces the condition $w_{\dot\alpha} = \bar{w}_{\dot\alpha} = 0$. Therefore the instanton induced terms to be discussed later vanish in this limit. This is in accord with the fact that the instanton induced terms have higher derivatives and vanish for dimensional reasons in the $\alpha' \to 0$ limit. We are here interested in a more general situation in which the gravitational sector does not decouple.

3.2. Broken and unbroken supersymmetries

The combined D-instanton/D3-brane system has bosonic zero modes associated with broken translation symmetries. These comprise the ten modes corresponding to the overall space-time translations of the composite system together with four zero modes that come from the invariance of the system under shifts of the D-instanton alone in the directions parallel to the D3-brane. Correspondingly a fraction of the thirty-two supersymmetry components are broken in this background. Sixteen of these broken supersymmetry components are superpartners of the broken overall translational symmetry. A further eight broken supersymmetry components are superpartners of the remaining four broken translations.

These statements are encoded in the supersymmetry algebra starting from the ten-dimensional Maxwell supersymmetry transformations, given by

$$\begin{aligned} \delta A_\mu &= i\bar{\eta}\Gamma_\mu\Psi, \\ \delta\Psi &= -\Gamma^{\mu\nu}F_{\mu\nu}\eta + \epsilon, \end{aligned} \tag{3.12}$$

where η and ϵ are two sixteen-component Majorana–Weyl spinors. After reduction to $p+1$ dimensions this algebra describes the supersymmetries of a Dp-brane background. The parameter η labels the sixteen unbroken supersymmetries while ϵ corresponds to the sixteen zero modes generated by the action of the broken supercharges on the background. The latter are the superpartners of the translational zero modes.

In the absence of the D-instanton the D3-brane is invariant under the action of $\mathcal{N}=4$ supersymmetry, which has $SU(4)$ as its R-symmetry group. Although the fully supersymmetric generalization of the Born–Infeld action is known [36,37,38] we need only consider the transformation of the low energy Maxwell system here. The action of the supersymmetries on the fields is given by the familiar $\mathcal{N}=4$ algebra,

$$\begin{aligned}
\delta\bar{\varphi}^{AB} &= \frac{1}{2}(\Lambda^{\alpha\,A}\eta^B_\alpha - \Lambda^{\alpha\,B}\eta^A_\alpha) + \frac{1}{2}\varepsilon^{ABCD}\overline{\xi}_{\dot\alpha\,C}\overline{\Lambda}^{\dot\alpha}_D \\
\delta\Lambda^A_\alpha &= -\frac{1}{2}F^-_{mn}\sigma^{mn}{}_\alpha{}^\beta\eta^A_\beta + 4i\,\partial_{\alpha\dot\alpha}\bar{\varphi}^{AB}\,\overline{\xi}^{\dot\alpha}_B \\
\delta A_m &= -i\Lambda^{\alpha\,A}\sigma^m{}_{\alpha\dot\alpha}\overline{\xi}^{\dot\alpha}_A - i\eta^{\alpha\,A}\sigma^m{}_{\alpha\dot\alpha}\overline{\Lambda}^{\dot\alpha}_A \\
\delta\bar{\Lambda}^{\dot\alpha}_A &= -\frac{1}{2}F^+_{mn}\bar{\sigma}^{mn}{}_{\dot\beta}{}^{\dot\alpha}\bar{\xi}^{\dot\beta}_A - 4i\,\partial^{\dot\alpha\alpha}\varphi_{AB}\,\eta^B_\alpha\,,
\end{aligned} \tag{3.13}$$

where lower indices $A,B=1,2,3,4$ label the defining representation of $SU(4)$ or, equivalently, a chiral spinor representation of $SO(6)$ (while upper indices denote the conjugate representations). The Grassmann variables $\bar{\xi}^{\dot\alpha}_A$ and η^A_α parameterize the sixteen unbroken supersymmetries that descend from η in (3.12) and are chiral spinors of both $SO(6)$ and $SO(4)$,

$$\frac{1}{\sqrt{2\pi}}\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\ \bar{\xi}^{\dot\alpha}_A\end{pmatrix}, \qquad \frac{1}{\sqrt{2\pi}}\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}\eta^A_\alpha\\ 0\end{pmatrix}, \tag{3.14}$$

where $SO(4)$ chirality is denoted, as usual, by dotted and undotted indices.

Likewise, in the absence of the D3-brane, the D-instanton is invariant under the sixteen supersymmetry transformations,

$$\begin{aligned}
\delta\bar{\chi}^{AB} &= \frac{1}{2}(M'^{\alpha\,A}\xi^B_\alpha - M'^{\alpha\,B}\xi^A_\alpha) + \frac{1}{2}\varepsilon^{ABCD}\overline{\xi}_{\dot\alpha}\,C\lambda^{\dot\alpha}_D \\
\delta a_m &= -iM'^{\alpha\,A}\sigma^m{}_{\alpha\dot\alpha}\overline{\xi}^{\dot\alpha}_A - i\xi^{\alpha\,A}\sigma^m{}_{\alpha\dot\alpha}\lambda^{\dot\alpha}_A,
\end{aligned} \tag{3.15}$$

where $\bar{\xi}^{\dot\alpha}_A$ and ξ^A_α are the sixteen supersymmetry parameters. The notation has been chosen to emphasize the fact that in the coupled system of a D3-brane in an instanton background eight of the supersymmetries in (3.13) and (3.15) are common and are therefore unbroken in the composite system. These are the ones associated with the parameters $\bar{\xi}^{\dot\alpha}_A$. The unbroken $(0,1)$ supersymmetry acts on the w and μ by the transformations

$$\begin{aligned}
\delta w_{\dot\alpha} &= \bar{\xi}_{A\dot\alpha}\mu^A \\
\delta\mu^A &= -4iw_{\dot\alpha}\bar{\chi}^{AB}\,\bar{\xi}^{\dot\alpha}_B \\
\delta\lambda^{\dot\alpha}_A &= \frac{i}{2}g_0^2\,\bar{w}w\,\bar{\xi}^{\dot\alpha}_A,
\end{aligned} \tag{3.16}$$

in addition to the $\bar{\xi}$ transformations in (3.13) and (3.15). As will be discussed later, in the presence of D3-brane source fields there are other $\bar{\xi}$ transformations that modify the transformations in (3.13).

The other twenty-four supersymmetry components are broken on one or other of the D-branes or on both. The quantity M' is the goldstino for the supersymmetry that is broken on the D-instanton and is associated with the shift transformation,

$$\delta_\eta {M'}^A_\alpha = \eta^A_\alpha, \tag{3.17}$$

with the remaining D-instanton coordinates and twist fields being inert. The field Λ is the goldstino for the supersymmetry that is broken on the D3-brane but not on the D-instanton which is implemented by the shift,

$$\delta_\xi \Lambda^A_\alpha = \xi^A_\alpha, \tag{3.18}$$

leaving w and μ and the other D3-brane fields inert. The final set of broken supersymmetries are broken on both the D3-brane and on the D-instanton and are associated with the transformations,

$$\delta_\rho \lambda^{\dot\alpha}_A = \rho^{\dot\alpha}_A = \delta\bar{\Lambda}^{\dot\alpha}_A, \tag{3.19}$$

where $\rho^{\dot\alpha}_A$ are eight further Grassmann variables. None of the other fields transform under these supersymmetry components.

The following table summarizes the various broken and unbroken supersymmetries (denoted b and u, respectively),

$$\begin{array}{ccccc} & \bar{\xi}_{\dot\alpha A} & \xi^A_\alpha & \rho_{\dot\alpha A} & \eta^A_\alpha \\ D3 & u & b & b & u \\ D(-1) & u & u & b & b \end{array}\ . \tag{3.20}$$

The twenty-four Grassmann parameters for the broken supersymmetries are the supermoduli of the system.

3.3. Vertex Operators and disk diagrams

The interactions between the open strings summarized by (3.11) can be obtained by considering the insertion of the various vertex operators on a disk which has a segment of purely Dirichlet boundary (the D segment) and a sector with Neumann conditions in the $m = 1, 2, 3, 4$ directions (the N segment). These boundary conditions require the presence of two twist operators that reside at the points at which the boundary conditions flip. This will later be seen to be a very useful way of summarizing the interactions of external D3-brane open-string sources.

The vertex operators attached to a Dirichlet boundary are those for the moduli of an isolated D-instanton and represent the interactions of open strings with endpoints satisfying purely Dirichlet (DD) boundary conditions.[5] They are given (in the -1 picture) by

$$V_{-1}(\chi^a) = \chi_a \, e^{-\phi}\psi^a, \qquad V_{-1}(a_n) = a_n \, e^{-\phi}\psi^n, \tag{3.21}$$

where here and in the following the subscript of the vertex operator denotes its superghost number. The vertex operators for the fermionic fields are (in the $-1/2$ picture)

$$V_{-1/2}(\lambda) = \lambda^{\dot\alpha}_A \, e^{-\phi/2}\Sigma_{\dot\alpha}\Sigma^A, \qquad V_{-1/2}(M') = M'^{\,A}_{\alpha} \, e^{-\phi/2}\Sigma^\alpha\Sigma_A, \tag{3.22}$$

where a $SO(10)$ spin field Σ^a $a = 1, \cdots, 16$ is decomposed into a product of a $SO(4)$ spin fields $\Sigma_a, \Sigma_{\dot a}$ and a $SO(6) = SU(4)$ spin fields Σ_A, Σ^A.

There are no interactions involving the strings that begin and end on the D-instanton. The terms in (3.11) arise from the disk with a portion of the boundary having Neumann conditions in the transverse directions and a portion having Dirichlet conditions. With these conditions two of the vertex operators must be twist operators. In the presence of a D3-brane there are additional collective coordinates which come from the strings stretched between the D3-brane and the D-instanton which have Neumann conditions in the $m = 0, 1, 2, 3$ directions at one end and Dirichlet in all other directions (DN strings) These descend from the hypermultiplets in six dimensions. The vertex operators for $w_{\dot\alpha}$ and μ^A are given (in the ghost number -1 picture) by

$$V_{-1}(w) = w_{\dot\alpha} e^{-\phi}\Delta\Sigma^{\dot\alpha}, \qquad V_{-1/2}(\mu) = \mu^A e^{-\phi/2}\Delta\Sigma_A, \tag{3.23}$$

where $\Delta = \sigma_0\sigma_1\sigma_2\sigma_4$ is the product of Z_2 twist fields which twists the bosonic field X^μ so that it interpolates between Neumann and Dirichlet boundary conditions in the $\mu = 0, 1, 2, 3$ directions.

The various terms in (3.11) follow simply by evaluating three-point and four-point functions of vertex operators on the disk, ensuring that the total superconformal ghost number is always -2. For example,

$$\langle cV(\lambda)cV(w)cV(\bar\mu)\rangle = i\pi \, \bar\mu^A w_{\dot\alpha}\lambda^{\dot\alpha}_A, \tag{3.24}$$

where c is the superconformal ghost. The other terms in (3.11) follow in a similar manner.

In addition to fields localized at the instanton there are open-string excitations on the three-brane world-volume. The massless bosonic excitations living on the three-brane

[5] From now on we will mainly use language of the D-instanton/D3-brane system although we will also make reference to the related D5-brane/D9-brane system.

and carrying longitudinal momentum k^m are described by vertex operators in the -1 superghost picture of the form

$$V_{-1}(A) = A_n e^{-\phi}\psi^n e^{ik\cdot X}, \qquad V_{-1}(\varphi) = \varphi_a e^{-\phi}\psi^a e^{ik\cdot X}, \tag{3.25}$$

while the gaugino vertex operators in the $-1/2$ picture are given by

$$V_{-1/2}(\bar\Lambda) = \bar\Lambda^{\dot\alpha}_A\, e^{-\phi/2}\Sigma_{\dot\alpha}\Sigma^A\, e^{ik\cdot X}, \qquad V_{-1/2}(\Lambda) = \Lambda^A_\alpha\, e^{-\phi/2}\Sigma^\alpha\Sigma_A\, e^{ik\cdot X}. \tag{3.26}$$

4. Inclusion of D3-brane sources

We would now like to generalize the moduli space action (3.11) to include sources that correspond to the couplings of the open-string ground states of the D3-brane. The resulting action should encode the supersymmetry transformations (3.12)-(3.19) and will lead to an evaluation of the leading effects of the D-instanton on open-string scattering on the D3-brane. In order to include these sources we need to consider the insertion of vertex operators on the N segment of the boundary of the disk. Such an operator describes an on-shell plane-wave scattering state with momentum k^m.

The simplest diagram of this type is given by the insertion a three-brane gauge field vertex and two bosonic twist fields without any M' insertions,

$$\langle F^+\rangle_{w\bar w} = \langle cV_0(A_m)cV_{-1}(\bar w)cV_{-1}(w)\rangle = W^{mn}\,F^+_{mn} \equiv W^c\bar\eta_c^{mn}F^+_{mn}, \tag{4.1}$$

which is proportional to the self-dual part of the Maxwell field. The wave function F^+ includes a plane-wave factor $e^{ik\cdot a}$, where the collective coordinate a^m will later be integrated. This is the basic one-point function from which other one-point functions for massless D3-brane states can be obtained by considering processes involving a single D3-brane vertex operator $V_g(\Phi)$ (where Φ is any D3-brane open-string ground state and g is the superghost number of the vertex) inserted onto a disk with two bosonic twist operators and with n insertions of integrated M' vertex operators on the D segment of the boundary. We will denote this process by the symbol

$$\langle\Phi\rangle_{w\bar w;n} = \langle cV_g(\Phi)cV_{-1}(\bar w)cV_{-1}(w)\int_D V_{l_1}(M')\ldots V_{l_n}(M')\rangle, \tag{4.2}$$

where $\int_D$ indicates that the integration is over the Dirichlet sector of the boundary and $l_r = \pm 1/2$ are the ghost numbers of the M' vertex operators. Since the two $V(w)$ vertices saturate the ghost number anomaly for the disk the total ghost number for the other vertex operators in (4.2) has to be zero, so that $g + \sum_{r=1}^n l_r = 0$. The quantity (4.2) is easily

evaluated in terms of the expression (4.1) as follows from the transformation properties of the D3-brane vertices under the η supersymmetries,

$$\begin{aligned} V_0(\delta_\eta A_m) =&[\eta Q_{+1/2}, V_{-1/2}(\bar\Lambda)], \qquad V_{-1/2}(\delta_\eta \bar\Lambda) = [\eta Q_{-1/2}, V_0(\partial\varphi)], \\ V_0(\delta_\eta \varphi) =&[\eta Q_{+1/2}, V_{-1/2}(\Lambda)], \qquad V_{-1/2}(\delta_\eta \Lambda) = [\eta Q_{-1/2}, V_0(F^-)], \end{aligned} \tag{4.3}$$

where the quantities $\delta_\eta\Phi$ are the η supersymmetry variations of (3.13). The supersymmetry charge Q can be represented as $\eta Q_{\pm 1/2} = \int_D V_{\pm 1/2}(\eta)$, where $V_{\pm 1/2}$ is the same expression as the M' vertex operator.

This can be used, for example, to evaluate the diagram with one $\bar\Lambda$ vertex, two bosonic twist fields and one M' vertex,

$$\begin{aligned} \langle \bar\Lambda\rangle_{w\bar w;1} &= \langle cV(\delta_{M'}A_m)cV(\bar w)cV(w)\rangle \\ &= \langle cV(\bar\Lambda)cV(\bar w)cV(w)\int_D V(M')\rangle \\ &= -iW^{mn}\, {M'}^A\sigma_{[m}\partial_{n]}\bar\Lambda_A. \end{aligned} \tag{4.4}$$

Successive application of the supersymmetry transformations parameterized by η gives

$$\begin{aligned} \langle\varphi_{AB}\rangle_{w\bar w;2} &= 4W^{mn}\ {M'}^B\sigma_{mp}{M'}^A\partial_n\partial^p\varphi_{AB} \\ \langle\Lambda\rangle_{w\bar w;3} &= 2W^{mn}\ \epsilon_{ABCD}{M'}^B\sigma_{pm}{M'}^A{M'}^{C\,\alpha}\partial_n\partial^p\Lambda^D_\alpha \\ \langle A\rangle_{w\bar w;4} &= W^{mn}\ \epsilon_{ABCD}{M'}^B\sigma_{pm}{M'}^A\ {M'}^C\sigma^{kl}{M'}^D\ \partial_n\partial^p F^-_{kl}. \end{aligned} \tag{4.5}$$

The collection of terms (4.1), (4.4) and (4.5) can be combined into a superfield, $\Phi_{mn}(x^n, M')W^{mn}$ where,

$$\Phi_{mn}(x^n, M') = F^+_{mn} + i{M'}^A\sigma_{[m}\,\partial_{n]}\bar\Lambda_A + \ldots. \tag{4.6}$$

The complete expansion for this superfield is given in section 4.1. In a similar manner it is easy to construct the disk amplitudes with two fermionic twist operators, μ and $\bar\mu$, which combine into the expression $\Phi_{AB}\bar\mu^A\mu^B$ where,

$$\Phi_{AB}(x^n, M') = \varphi_{AB} + \frac{1}{2}\epsilon_{ABCD}{M'}^{[C\alpha}\,\Lambda^{D]}_\alpha + \ldots. \tag{4.7}$$

Finally, the disk diagrams with one bosonic and one fermionic twist vertex combine into $\Phi^{\dot\alpha}_A(x^n, M')w_{\dot\alpha}\mu^A$ where,

$$\Phi^{\dot\alpha}_A(x^n, M') = \bar\Lambda^{\dot\alpha}_A - 4i{M'}^B_\beta\,\sigma^{n\,\beta\dot\alpha}\,\partial_n\varphi_{AB} + \ldots. \tag{4.8}$$

The complete expressions for $\Phi_{AB}(x^n, M')$ and $\Phi^{\dot\alpha}_A(x^n, M')$ are also given in section 4.1. By construction, these superfields are invariant under the supersymmetry transformations generated by shifts of $M' \to M' + \eta$,

$$\delta_\eta \Phi = \eta^A_\alpha \frac{\partial}{\partial M'^A_\alpha} \Phi, \tag{4.9}$$

provided the component fields transform under the eight η supersymmetry transformations of (3.13).

The complete moduli space action, including the D3-brane sources, can now be written as

$$\begin{aligned} S_{-1}[\Phi_{mn}, \Phi^{\dot\alpha}_A, \Phi_{AB}] = &- 2\pi i\tau + \frac{1}{4} g_0^2 (W^c)^2 + 2(\chi_{AB} - \Phi_{AB})^2 W_0 - i\Phi_{mn}\bar\eta^c_{mn} W^c \\ &- 2i(\chi_{AB} - \Phi_{AB})\bar\mu^A \mu^B - i(\lambda^{\dot\alpha}_A - \Phi^{\dot\alpha}_A)(\bar\mu^A w_{\dot\alpha} + \mu^A \bar w_{\dot\alpha}), \end{aligned} \tag{4.10}$$

so that $S_{-1}[0,0,0] = S_{-1}$. The dimensions of all the fields in this expression are determined by the fact that g_0 is proportional to ${\alpha'}^{-1}$. In particular, this is consistent with the assignment of the canonical dimensions to the free four-dimensional bosonic and fermionic fields associated with the open strings of the D3-brane which are the leading terms in the expansion of the superfields Φ_{mn}, $\Phi^{\dot\alpha}_A$ and Φ_{AB}.

4.1. Supersymmetries in the presence of D3-brane sources

The action (4.10) is invariant under all thirty-two supersymmetries provided the component D3-brane fields transform in the appropriate manner. These are the eight conserved supersymmetries with parameter $\bar\xi^{\dot\alpha}_A$ and the twenty-four supersymmetries with parameters η^A_α, $\rho^{\dot\alpha}_A$ and ξ^A_α. In order to analyze this more completely it is useful to express the sources in terms of $N = 4$ on-shell superfields.

Recall [39] that the physical $\mathcal{N} = 4$ fields satisfying the linearized equations of motion are contained in a superfield $\mathcal{W}_{AB}(\theta, \bar\theta)$, where the sixteen Grassmann superspace parameters are components of θ^A_α and $\bar\theta^{\dot\alpha}_A$ which transform in the $(\mathbf{4}, \mathbf{2}, \mathbf{1})$ and $(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})$ of $SU(4) \otimes SU(2)_L \otimes SU(2)_R$, respectively. Covariant derivatives $\bar D^A_\alpha$ and D^α_A are defined by

$$D^A_{\dot\alpha} = -\frac{\partial}{\partial \bar\theta^{\dot\alpha}_A} - i(\sigma\cdot\partial)_{\dot\alpha\beta}\theta^{A\beta}, \qquad \bar D^\alpha_A = \frac{\partial}{\partial \theta^A_\alpha} + i(\sigma\cdot\partial)_{\alpha\dot\beta}\bar\theta^{A\dot\beta}, \tag{4.11}$$

while the supersymmetries are represented as

$$Q^A_{\dot\alpha} = -\frac{\partial}{\partial \bar\theta^{\dot\alpha}_A} + i(\sigma\cdot\partial)_{\dot\alpha\beta}\theta^{A\beta}, \qquad \bar Q^\alpha_A = \frac{\partial}{\partial \theta^A_\alpha} - i(\sigma\cdot\partial)_{\alpha\dot\beta}\bar\theta^{A\dot\beta}, \tag{4.12}$$

The bi-fundamental superfield is defined to satisfy the constraints,

$$\mathcal{W}_{AB} = -\mathcal{W}_{BA} = \frac{1}{2}\epsilon_{ABCD}\bar{\mathcal{W}}^{CD}, \qquad D^C_{\dot\alpha}\mathcal{W}_{AB} = \delta^C_{[A}\mathcal{W}_{B]\dot\alpha}. \tag{4.13}$$

It follows that the first few terms in the expansion of this superfield have the form

$$\begin{aligned}\mathcal{W}_{AB} = \varphi_{AB} + \bar\theta_{[A\dot\alpha}\,\bar\Lambda^{\dot\alpha}_{B]} + \frac{1}{2}\epsilon_{ABCD}\theta^{\alpha[C}\,\Lambda^{D]}_{\alpha} - \frac{1}{4}\epsilon_{ABCD}\theta^C\sigma^{mn}\theta^D\,F^-_{mn} \\ - \frac{1}{2}\bar\theta_A\bar\sigma^{mn}\bar\theta_B\,F^+_{mn} + i\theta^C\sigma^m\bar\theta_C\,\partial_m\varphi_{AB} + \dots .\end{aligned} \tag{4.14}$$

The derivative superfield, $\mathcal{W}^{\dot\alpha}_B$, is defined in (4.13) and can be written as a covariant derivative on $\mathcal{W}_{AB}$,

$$\mathcal{W}^{\dot\alpha}_B = \frac{3}{2}D^{A\dot\alpha}\mathcal{W}_{AB}. \tag{4.15}$$

Similarly, a tensor superfield $\mathcal{W}_{mn}$ is obtained by applying a further covariant derivative,

$$\mathcal{W}_{mn} = D^A\bar\sigma_{mn}D^B\,\mathcal{W}_{AB}. \tag{4.16}$$

The D3-brane fields that enter in the action S_{-1} are functions of ${M'}^A_\alpha$ that are identified as follows,

$$\Phi_{mn} = \mathcal{W}_{mn}|_{\bar\theta=0,\theta=M'}, \qquad \Phi^{\dot\alpha}_A = \mathcal{W}^{\dot\alpha}_A|_{\bar\theta=0,\theta=M'}, \qquad \Phi_{AB} = \mathcal{W}_{AB}|_{\bar\theta=0,\theta=M'}. \tag{4.17}$$

The fact that these are the correct identifications follows from the fact that they transform appropriately under the η supersymmetry transformations, which are given by $\eta^A_\alpha\bar Q^\alpha_A\,\Phi = -\eta^A_\alpha(\partial/\partial{M'}^A_\alpha)\,\Phi$.

The $\bar\xi$ supersymmetry transformation of Φ_{AB} is given by

$$\begin{aligned}\delta_{\bar\xi}\Phi_{AB} = \bar\xi_C\,Q^C\,\mathcal{W}_{AB}|_{\bar\theta=0,\theta=M'} &= \bar\xi_C\,D^C\,\mathcal{W}_{AB}|_{\bar\theta=0,\theta=M'} - 2\bar\xi_C\sigma\cdot\partial\theta^C\,\Phi_{AB} \\ &= \bar\xi_{[A}\Phi_{B]} - 2\bar\xi_C\sigma\cdot\partial\theta^C\,\Phi_{AB},\end{aligned} \tag{4.18}$$

while the $\bar\xi$ supersymmetry transformation of $\Phi^{\dot\alpha}_A$ is given by

$$\delta_{\bar\xi}\Phi^{\dot\alpha}_A = \bar\xi_C\,Q^C\,\mathcal{W}^{\dot\alpha}_A|_{\bar\theta=0,\theta=M'} = \bar\sigma^{mn}_{\dot\alpha\dot\beta}\bar\xi^{\dot\beta}_A\Phi_{mn} - 2\bar\xi_C\sigma\cdot\partial\theta^C\,\Phi^{\dot\alpha}_A, \tag{4.19}$$

and the transformation of Φ_{mn} is given by

$$\delta_{\bar\xi}\Phi_{mn} = \bar\xi_C Q^C\Phi_{mn} = -2\bar\xi_C\sigma\cdot\partial\theta^C\,\Phi_{mn}. \tag{4.20}$$

The $\bar\xi_C\sigma\cdot\partial\theta^C$ terms are proportional to the momentum carried by the source and will not contribute to the supersymmetry variation of any amplitude since the sum of the momenta carried by the sources vanishes. We may therefore drop the second terms on the right-hand

sides of (4.19) and (4.18). The η supersymmetry transformations act as shifts of M', as can see from the explicit expression,

$$\delta_\eta \Phi_{AB} = \eta^C_\alpha \bar{Q}^\alpha_C \mathcal{W}_{AB}|_{\bar\theta=0,\theta=M'} = \eta^C_\alpha \frac{\partial}{\partial {M'}^C_\alpha}\,\Phi_{AB}. \tag{4.21}$$

This accounts for eight of the nonlinearly realized supersymmetries of the collective coordinate action.

The expansions for the source superfields terminate with the terms of order ${M'}^4$ upon using the equations of motion and have the explicit form,

$$\begin{aligned}\Phi_{mn}(x^n, M') =& F^+_{mn} + i{M'}^A \sigma_{[m}\,\partial_{n]}\bar\Lambda_A + 4{M'}^B \sigma_{[m}{}^p {M'}^A \partial_{n]}\partial_p \varphi_{AB}\\ &+ 2\epsilon_{ABCD}{M'}^B\sigma_{p[m}{M'}^A{M'}^C\partial_{n]}\partial_p\Lambda\\ &+ \epsilon_{ABCD}{M'}^B\sigma_{p[m}{M'}^A\;{M'}^C\sigma^{kl}{M'}^D\;\partial_{n]}\partial_p F^-_{kl},\end{aligned}\tag{4.22}$$

$$\begin{aligned}\Phi^{\dot\alpha}_A(x^n, M') =& \bar\Lambda^{\dot\alpha}_A - 4i{M'}^B_\beta\,\sigma^{n\,\beta\dot\alpha}\,\partial_n\varphi_{AB} - 2i\epsilon_{ABCD}{M'}^B_\beta\,\sigma^{n\,\beta\dot\alpha}\;{M'}^C\partial_n\Lambda^D\\ &+ i\epsilon_{ABCD}{M'}^B_\beta\,\sigma^{n\,\beta\dot\alpha}\;{M'}^C\sigma^{pq}{M'}^D\partial_n F^-_{pq}\\ &+ \epsilon_{ABCD}{M'}^B_\beta\,\sigma^{n\,\beta\dot\alpha}\;{M'}^C\sigma^{pq}{M'}^D\partial_n\partial_{[p}{M'}^E\sigma_{[q}\partial_n\partial_{p]}\bar\Lambda_E,\end{aligned}\tag{4.23}$$

$$\begin{aligned}\Phi_{AB}(x^n, M') =& \varphi_{AB} + \frac{1}{2}\epsilon_{ABCD}{M'}^{[C\alpha}\,\Lambda^{D]}_\alpha - \frac{1}{4}\epsilon_{ABCD}{M'}^C\sigma^{mn}{M'}^D\,F^-_{mn}\\ &+ \frac{i}{4}\epsilon_{ABCD}{M'}^C\sigma^{mn}{M'}^D\;{M'}^E\sigma_{[m}\partial_{n]}\bar\Lambda_E\\ &+ \epsilon_{ABCD}{M'}^C\sigma^{mn}{M'}^D\;{M'}^E\sigma_{pm}{M'}^F\partial_n\partial^p\varphi_{EF}.\end{aligned}\tag{4.24}$$

The action (4.10) only involves the relative superfields,

$$\hat\Phi_{AB} = \Phi_{AB} - \chi_{AB}, \qquad \hat\Phi_{A\dot\alpha} = \Phi_{A\dot\alpha} - \lambda_{A\dot\alpha}. \tag{4.25}$$

Consequently, in the presence of open-string sources the $\bar\xi$ transformation of μ in (3.16) is replaced by the translationally-invariant expression,

$$\delta_{\bar\xi}\mu^A = -4iw_{\dot\alpha}\bar{\hat\Phi}^{AB}\,\bar\xi^{\dot\alpha}_B, \tag{4.26}$$

while $\delta_{\bar\xi}w$ remains unchanged. The $\bar\xi$ variation of the action (4.10) is given by,

$$\begin{aligned}\delta_{\bar\xi}S_{-1}[\Phi_{mn},\Phi^A_{\dot\alpha},\Phi_{AB}] =& -i\delta_{\bar\xi}\Phi_{mn}\bar\eta^c_{mn}W^c + 2i\delta_{\bar\xi}\hat\Phi_{AB}\bar\mu^A\mu^B + i\delta_{\bar\xi}\hat\Phi^{\dot\alpha}_A(\bar\mu^A w_{\dot\alpha} + \mu^A\bar w_{\dot\alpha})\\ &+ 4\delta_{\bar\xi}\hat\Phi_{AB}\,\bar{\hat\Phi}^{AB}\,W_0 + (\tfrac{1}{2}g_0^2W_0 + 2\hat\Phi^2_{BC})\,(\bar w\bar\xi_A\,\mu^A + w\bar\xi_A\,\bar\mu^A)\\ &- i\Phi_{mn}(\bar w\bar\sigma^{mn}\bar\xi_A\,\mu^A - \bar\mu^A\,\bar\xi_A\bar\sigma^{mn}w)\\ &+ i\hat\Phi_{A\dot\alpha}(\bar\mu^A\bar\xi^{\dot\alpha}_B\mu^B + \mu^A\bar\xi^{\dot\alpha}_B\bar\mu^B - 4i\bar w^{\dot\alpha}\bar{\hat\Phi}^{AB}\,\bar\xi_B w - 4iw^{\dot\alpha}\bar{\hat\Phi}^{AB}\,\bar\xi_B\bar w)\\ &+ 8\hat\Phi_{AB}(\bar\mu^A\,w\bar{\hat\Phi}^{BC}\bar\xi_C + \bar w\bar{\hat\Phi}^{AC}\bar\xi_C\,\mu^B).\end{aligned}\tag{4.27}$$

The requirement that $\delta_{\bar\xi} S_{-1}[\Phi_{mn}, \Phi^A_{\dot\alpha}, \Phi_{AB}] = 0$ determines the transformations of the D3-brane source superfields which are given by

$$\delta_{\bar\xi}\Phi_{mn} = 0, \qquad \delta_{\bar\xi}\hat\Phi^{\dot\alpha}_A = \bar\sigma^{mn}_{\dot\alpha\dot\beta}\bar\xi^{\dot\beta}_A\big(\Phi_{mn} + \frac{i}{2}g_0^2 W_{mn}\big), \qquad \delta_{\bar\xi}\hat\Phi_{AB} = \bar\xi_{[A}\Phi_{B]}, \tag{4.28}$$

where the D3-brane fields are evaluated at $x^m = a^m$. These transformations differ from the $\bar\xi$ transformations of the $\mathcal{N} = 4$ theory only by the term proportional to g_0^2.

5. Integration over collective coordinates

The effect of the D-instanton on open-string scattering amplitudes is obtained by the integration over the collective coordinates, which is weighted by the instanton action (4.10) with D3-brane source terms included.

$$Z[\Phi_{mn}, \Phi^{\dot\alpha}_A, \Phi_{AB}] = \mathcal{C}\int d^8M'\, d^8\lambda\, d^4\mu\, d^4\bar\mu\, d^4a\, d^6\chi\, d^2w\, d^2\bar w\, e^{-S_{-1}[\Phi_{mn},\Phi^{\dot\alpha}_A,\Phi_{AB}]}. \tag{5.1}$$

The nomalization $\mathcal{C}$ will only be determined up to an overall numerical constant, c, although the dependence on α' and the string coupling, e^ϕ, is important in the following and will be displayed. This normalization has the form

$$\mathcal{C} = c\,\times g_0^3 \times g_4^4, \tag{5.2}$$

where g_{p+1} is defined in (2.4). The factor of g_0^3 comes from integrating out the auxiliary field D_c and the factor of g_4^4 comes from the normalization of the instanton collective coordinate measure [18] for $N = 1, k = 1$.

5.1. Integration over fermionic collective coordinates

Integration over the fermionic coordinates only gives a non-zero result when there is an appropriate number of insertions of fermionic sources. We will consider the μ integrations first. These can be performed in several ways, giving rise to distinct sectors according to how many factors of λ are brought down from the expansion of $e^{-S_{-1}}$. For example, one way to perform the μ integrations is to bring down four factors of $\bar\mu^A w_{\dot\alpha}\lambda^{\dot\alpha}_A$ and four factors of $\bar w_{\dot\alpha}\mu^A\lambda^{\dot\alpha}_A$ from $e^{-S_{-1}}$. In that case the combined μ, λ integrations enter in the form,

$$\int d^8\lambda d^4\mu d^4\bar\mu \exp\left(i\bar\mu^A w_{\dot\alpha}\lambda^{\dot\alpha}_A + i\bar w_{\dot\alpha}\mu^A\lambda^{\dot\alpha}_A\right) = (\bar w^{\dot\alpha}w_{\dot\alpha})^4 = W_0^4. \tag{5.3}$$

We will call this sector, in which all eight powers of λ are soaked up by the μ integration, the 'minimal' sector. More generally, there are sectors in which factors of $(\chi_{AB} - \Phi_{AB})\bar\mu^A\mu^B$ or $\Phi^{\dot\alpha}_A\,\bar\mu^A w_{\dot\alpha}$ are brought down from the expansion of $e^{-S_{-1}[\Phi_{mn},\Phi^A_{\dot\alpha}\Phi_{AB}]}$. In such cases the integration over μ brings down less than eight powers of λ, resulting in 2, 4, 6 or 8

unsaturated components of λ. Since there are no open-string D3-brane sources that couple to λ, these sectors of the integral will vanish in the absence of closed-string sources. We will see later that closed-string sources couple to a disk with a number of λ vertex operators (and a pair of twist operators) attached so these can provide the missing powers of λ. However, it will still be true that the instanton induced interactions of lowest dimension arise in the minimal sector.

For the moment we will consider case in which there are no closed-string sources so that only the minimal sector is relevant. The integration over M' necessarily brings down powers of the open-string sources since M' does not appear in the source-free action, $S_{-1}[0,0,0]$ (4.10). The non-zero instanton-induced amplitudes that come from expanding (5.1) in powers of the sources have the form (absorbing an overall constant into c),

$$A^{inst}_{\Phi_1\ldots\Phi_r} = \mathcal{C}\int d^4a d^2w d^2\bar{w} d^6\chi\, d^8M'\, W_0^4\, e^{-\hat{S}_{-1}}\, \langle\Phi_1\rangle_{w\bar{w};m_1}\ldots\langle\Phi_r\rangle_{w\bar{w};m_r}, \tag{5.4}$$

where $\sum_r m_r = 8$ and Φ_r are the component D3-brane fields that couple to $m_r\ V(M')$ vertex operators and two bosonic twist operators. The integration over the a^m coordinates leads to an overall momentum conservation delta function, $\delta^{(4)}(\sum_r k_r^m)$, which will be suppressed in the following formulae. The quantity $\hat{S}_{-1}$ in (5.4) is given by

$$\hat{S}_{-1} = -2\pi i\tau + \frac{1}{4}g_0^2 W_0^2 + \chi_a^2 W_0. \tag{5.5}$$

In order to perform the w and $\bar{w}$ integrations it will be convenient to change variables to W^c defined in (3.8). The integration measure for $w_{\dot{\alpha}}$ becomes

$$\int d^2w d^2\bar{w} = 2\pi\int \frac{dW^1 dW^2 dW^3}{W_0}, \tag{5.6}$$

where $W_0^2 = \sum_{c=1}^3 (W^c)^2$. Since W^c appears quadratically in the action the W^c integral is a simple Gaussian. The integration over the four bosonic moduli a^n, which imposes conservation of overall longitudinal momentum on the scattering amplitudes, has been performed in writing (5.4).

For definiteness we will now consider the case of four $\langle\varphi\rangle_{w\bar{w};2}$ insertions in detail. In that case the collective coordinate integration is given by

$$\begin{aligned} A^{inst}_{\varphi^4} &= \mathcal{C}\int \frac{d^3W^c}{W_0} d^6\chi d^8M'\, e^{-\hat{S}_{-1}}\, \langle\varphi\rangle_{w\bar{w};2}\,\langle\varphi\rangle_{w\bar{w};2}\,\langle\varphi\rangle_{w\bar{w};2}\,\langle\varphi\rangle_{w\bar{w};2} \\ &= \mathcal{C}\int \frac{d^3W^c}{W_0} d^6\chi d^8M'\, e^{-\hat{S}_{-1}}\, \bar{\eta}^{d_1}_{m_1n_1}\bar{\eta}^{d_2}_{m_2n_2}\bar{\eta}^{d_3}_{m_3n_3}\bar{\eta}^{d_4}_{m_4n_4} W^{d_1}W^{d_2}W^{d_3}W^{d_4} \\ &\times \eta^{c_1}_{p_1m_1}\,\eta^{c_2}_{p_2m_2}\,\eta^{c_3}_{p_3m_3}\,\eta^{c_4}_{p_4m_4} M'\Sigma^{a_1}\tau^{c_1}M'\partial_{n_1}\partial^{p_1}\varphi^{a_1}(x_1)\ M'\Sigma^{a_2}\tau^{c_2}M'\partial_{n_2}\partial^{p_2}\varphi^{a_2}(x_2) \\ &\times\ M'\Sigma^{a_3}\tau^{c_3}M'\partial_{n_3}\partial^{p_3}\varphi^{a_3}(x_3) M'\Sigma^{a_4}\tau^{c_4}M'\partial_{n_4}\partial^{p_4}\varphi^{a_4}(x_4), \end{aligned} \tag{5.7}$$

where we have used the relation between σ^{mn} and τ^c in appendix A to write

$$M'^A \sigma_{pm} M'^B \varphi_{AB}(x) = M' \Sigma^a \tau^c M' \, \varphi^a(x) \eta^c_{pm}. \tag{5.8}$$

It is convenient to work in momentum space so that $\partial^n = ik^n$ in (5.7), where the integration over a_n in (5.1) guarantees momentum conservation, $\sum_r k^n_r = 0$.

In contemplating the W^c integration it is useful to transform to polar coordinates (W_0, θ, ϕ), where the measure is

$$d^3 W^c W_0^{-1} = dW_0 \, W_0 \, \sin\theta d\theta d\phi, \tag{5.9}$$

and the angular part of the integral (5.7) gives,

$$\int \sin\theta d\theta \, d\phi \, W^{d_1} W^{d_2} W^{d_3} W^{d_4} = \frac{4\pi}{15} W_0^4 \left(\delta^{d_1 d_2}\delta^{d_3 d_4} + \delta^{d_1 d_3}\delta^{d_2 d_4} + \delta^{d_1 d_4}\delta^{d_2 d_3}\right), \tag{5.10}$$

where the radial W_0 variable will be integrated later.

In order to simplify the structure of the integrand we may make use of the identities involving products of 'tHooft's η symbols which follow from their definition (see appendix A). A very simple way to evaluate the fermionic integrations is given by choosing a particular frame for the momenta in which one component, k^0_r, of each momentum vanishes. In that case, after using the mass-shell condition ($k_r^2 = 0$) we have the equation,

$$\eta^{c_r}_{m_r p_r} \bar\eta^{d_r}_{m_r n_r} k^{p_r}_r k^{n_r}_r = 2 k^{c_r}_r k^{d_r}_r, \tag{5.11}$$

Using this expression in (5.7) together with the (5.10) leads to an expression for M' proportional to

$$(s^2 + t^2 + u^2) \int d^8 M' \left(\prod_{r=1}^{4} M' \Sigma^{a_r} \tau^{c_r} M' k^{c_r}_r \varphi^{a_r}(x_r) \right), \tag{5.12}$$

The factor of $(s^2 + t^2 + u^2)$ comes from the contractions of the momenta with the $\delta^{d_r d_s}$ factors in (5.10), which arose from the W^c integrations.

The integrand of (5.12) involves the fermion bilinears $M' \Sigma^{a_r} \tau^{c_r} M'$ (where $a_r = 1, \cdots, 5$ and $c_r = 1, \cdots, 3$) which have the same structure as the SO(8) spinor bilinears of light-cone superstring theory. A standard argument relates the integral over a product of the components of a fermionic $SO(8)$ spinor to the tensor t_8 in (2.7) so that the integration over M' produces

$$\begin{aligned} \int d^8 M' \left(\prod_{r=1}^{4} M' \Sigma^{a_r} \tau^{c_r} M' \, k^{c_r}_r \varphi^{a_r}(x_r) \right) &= t_8^{a_1 c_1 \ldots a_4 c_4} k^{c_1}_1 \varphi^{a_1}(x_1) \ldots k^{c_4}_4 \varphi^{a_4}(x_r) \\ &= tu \; \varphi(x_1) \cdot \varphi(x_2) \; \varphi(x_3) \cdot \varphi(x_4) + \text{ perms.} \end{aligned} \tag{5.13}$$

Hence the instanton induced $(\partial^2\varphi)^4$ term is given by

$$A^{inst}_{\varphi^4} = (s^2+t^2+u^2)\big(tu\varphi_1\cdot\varphi_2\varphi_3\cdot\varphi_4 + su\varphi_1\cdot\varphi_4\varphi_2\cdot\varphi_3 + st\varphi_1\cdot\varphi_3\varphi_2\cdot\varphi_4\big)\; I_4, \quad (5.14)$$

where I_4 denotes the remaining integrations over the bosonic collective coordinates, χ^a and W^c, which will be discussed in the next subsection. Note that (5.14) has the same form as the kinematic dependence of the tree-level diagram in (2.9). Although our derivation of this expression used a particular frame, a more careful analysis gives the same result, as is expected from Lorentz invariance.

In a similar manner one can discuss amplitudes for scattering any of the other open-string states. For example, the insertion of two $\langle F^+\rangle_{w\bar{w};4}$ sources saturates the M' integration, leading to a possible $\partial^2 F^+\partial^2 F^+$ term in the action. However this is a total derivative and vanishes after integration. At least two powers of the source $\langle F^-\rangle_{w\bar{w};0}$ have to be brought down from the exponential to produce a non vanishing result,

$$\begin{aligned}
A^{inst}_{F^4} =&\mathcal{C}\int \frac{d^3W^c}{W_0} d^6\chi d^8M'\, e^{-\hat{S}_{-1}}\,\langle F^+\rangle_{w\bar{w};0}\,\langle F^+\rangle_{w\bar{w};0}\,\langle F^-\rangle_{w\bar{w};4}\,\langle F^-\rangle_{w\bar{w};4}\\
=&\mathcal{C}\int \frac{d^3W^c}{W_0} d^6\chi d^8M'\, e^{-\hat{S}_{-1}}\bar{\eta}^{d_1}_{m_1n_1}\bar{\eta}^{d_2}_{m_2n_2}\bar{\eta}^{d_3}_{m_3n_3}\bar{\eta}^{d_4}_{m_4n_4}\\
&\times W^{d_1}W^{d_2}W^{d_3}W^{d_4}F^+_{m_1n_1}(x_1)F^+_{m_2n_2}(x_2)\\
&\times \epsilon_{A_3B_3C_3D_3}{M'}^{A_3}\sigma_{p_3m_3}{M'}^{B_3}{M'}^{C_3}\sigma^{k_3l_3}{M'}^{D_3}\;\partial_{n_3}\partial_{p_3}F^-_{k_3l_3}(x_3)\\
&\times \epsilon_{A_4B_4C_4D_4}{M'}^{A_4}\sigma_{p_4m_4}{M'}^{B_4}{M'}^{C_4}\sigma^{k_4l_4}{M'}^{D_4}\;\partial_{n_4}\partial_{p_4}F^-_{k_4l_4}(x_4).
\end{aligned} \quad (5.15)$$

The integration over M' and W^c produces (after integration by parts) a term of the form

$$s^2(F^+)^2(F^-)^2, \quad (5.16)$$

The full amplitude is given by summing over the two other choices for the location of the fermionic zero modes. Since $(F^+)^2(F^-)^2$ is invariant under permutations of the fields the result is

$$A^{inst}_{(F^+)^2(F^-)^2} = (s^2+t^2+u^2)(F^+)^2(F^-)^2 = (s^2+t^2+u^2)\big(F^4 - \frac{1}{4}(F^2)^2\big)\; I_4, \quad (5.17)$$

which is the same tensor structure as the perturbative contribution (2.13). A straightforward extension of this argument also gives the instanton-induced interaction between two scalars and two field strengths as $\partial F^-\partial F^+(\partial^2\varphi)^2$ which also agrees with the structure of the analogous tree-level term (2.13).

5.2. Integration over bosonic collective coordinates

In general we want to consider processes with r external D3-brane open-string fields in (5.4), where the case of $r = 4$ was considered in the previous subsection and we will consider $r = 6, 8$ in section 6. In order to accommodate all these cases we can rewrite the generating function (5.1) after performing the λ, μ and $\bar{\mu}$ integrals and in the absence of closed-string sources,[6] in the form,

$$Z[\Phi_{mn}] = \mathcal{C} \int d^4a^m \, d^6\chi^a \, d^8M' \, \frac{d^3W^c}{W_0} \, e^{-\hat{S}_{-1}} \, e^{-i\Phi_{mn}(M')W^{mn}}. \qquad (5.18)$$

This generates all the instanton-induced open-string amplitudes, which are obtained by expanding the exponential to extract the terms that are eighth order in M'. The modular invariant interactions (such as $(\partial F)^4$) come from the term in which four powers of W^{mn} are brought down from the exponential. As will be discussed in more detail in section 6, there are also interactions that transform with modular weight $(1, -1)$ (such as $(\partial^2\varphi)^2(\partial\bar{\Lambda})^4$) that arise from terms with six powers of W^{mn} as well as interactions that transform with weight $(2, -2)$ (such as $(\partial\bar{\Lambda})^8$) that come from terms with eight powers of W^{mn}. It is evident by a simple rescaling of W that for fixed χ the perturbative expansion of (5.18) is actually an expansion in powers of $g_0^2 L^{-4}$, where $L = |\chi^a|$. This corresponds to the genus expansion of the world-sheet configurations that contribute to the process. Every term in this series is singular in the $L = 0$ limit although the exact expression (5.18) is well-defined.

In order to evaluate the amplitudes for scattering D3-brane fields it is instructive to perform the χ and W^c integrations in (5.18) before performing the M' integrations. Transforming the W^c integration to polar coordinates and performing the gaussian χ^a integrals results in the expression,

$$Z[\Phi_{mn}] = (2\pi)^5 \mathcal{C} \int d^4a \, dW_0 \, \sin\theta d\theta \, W_0^2 \, \exp\left(-iW_0 \, \cos\theta \, |\Phi| - \frac{g_0^2}{4} W_0^2\right), \qquad (5.19)$$

where $|\Phi| = (\Phi_{mn}\Phi^{mn})^{1/2}$. After changing variables from W_0 to $\hat{W}_0 = g_0 W_0/2$ and performing the θ integration the result is (again absorbing an overall constant into c)

$$Z[\Phi_{mn}] = c g_4^4 \int d^8M' \int d^4a^m d\hat{W}_0 \, \hat{W}_0^2 \, \frac{\sin y}{y} \, e^{-\hat{W}_0^2} \equiv c \int d^8M' \sum_{r=even} I_r \, |\Phi|^r, \qquad (5.20)$$

where the da^n integral gives a suppressed factor of $\delta^{(4)}(\sum_{s=1}^r k_s^m)$ and

$$y = g_0^{-1} \, \hat{W}_0 \, |\Phi|. \qquad (5.21)$$

[6] We are still considering the 'minimal' case defined in the first paragraph of 5.2, which is the only sector of relevance in the absence of closed-string sources.

The various terms that enter into the expansion of the integrand to eighth order in M' are easily extracted from (5.20) by using the expansion

$$\frac{\sin y}{y} = \sum_{r=even}^{\infty} \frac{y^r}{(r+1)!}(-1)^{r/2}. \tag{5.22}$$

The $\hat{W}_0$ integration in (5.20) is contained in

$$\begin{aligned} I(e^{\phi}) &= g_4^4 c \int d\hat{W}_0\, \hat{W}_0^2\, e^{-\hat{W}_0^2} \frac{\sin y}{y} \\ &= \sum_{r=even} {\alpha'}^r {\tau_2}^{-2+r/2}\, I_r, \end{aligned} \tag{5.23}$$

where the coefficients I_r are given by

$$I_r = c\, 2^{4+r} \pi^{2+3r/2} \frac{\Gamma\left(\frac{r}{2}+\frac{3}{2}\right)}{\Gamma(r+2)} (-1)^{r/2}. \tag{5.24}$$

The gaussian $\hat{W}_0$ integration in each term in the expansion of the integrand of (5.20) peaks at $\hat{W}_0 = 0$ and $\langle \hat{W}_0 \rangle = 0$. However, if a non-zero background B_{mn} field is present the integrand in (5.18) is multiplied by $e^{-iB_{mn}W^{mn}/2\pi\alpha'}$ so that only the gauge invariant combination $B_{mn} + 2\pi\alpha' F_{mn}$ enters. In that case the W_0 integration peaks at $W_0 \sim |B_{mn}|/g_0^2\pi\alpha'$. The systematics of the induced interactions is then quite different since there are odd powers of y in the expansion in (5.23). Furthermore, as discussed in [40], a finite B background rotates the boundary conditions and changes the identification of the broken and unbroken supersymmetries. The case of non-zero B will not be discussed in this paper.

We see from (5.20) that the leading contributions to the D-instanton induced interactions in the $r = 4$ terms are independent of $\tau_2 = e^{-\phi}$ and their dependence on the string length is given by ${\alpha'}^4$. We also saw in section 5.1 that the ${M'}^8$ terms in the expansion of $|\Phi_{mn}|^4$ have the same tensor structure as the corresponding tree-level terms. These facts mean that we can identify the leading instanton contribution to the $r = 4$ terms with the instanton terms in the weak-coupling expansion of (2.22), using (1.2) (which has the property that the leading instanton contributions are actually exact). A more thorough treatment would also check the absolute normalization of the instanton contribution which we have not determined. The dependence of (5.1) on α' and τ_2 for terms with $r = 6$ is given by ${\alpha'}^6\tau_2$ and for the $r = 8$ terms by ${\alpha'}^8\tau_2^2$, which will be important for other open string interactions discussed in section 6.

6. Other open-string interactions

The interactions described in section 5 are ones in which the coupling constant appears in the modular invariant function $h(\tau,\bar\tau)$ defined in (1.1). More generally, there are interactions that transform with non-zero modular weight. The modular invariance of the D3-brane theory is best examined in the Einstein frame of the bulk theory with metric $g^{(E)} = \tau_2^{1/2} g^{(s)}$ or $e^{(E)} = \tau_2^{1/4} e^{(s)}$ (where the superscripts $^{(E)}$ and $^{(s)}$ denote the Einstein and string frame, respectively and $e \equiv e_{\mu\, m}$ is the vierbein), since the metric is $SL(2,Z)$ invariant in that frame. The conventional normalization of the $\mathcal{N}=4$ Maxwell action is obtained in this frame provided the Einstein-frame fields are related to those in the string frame in the following manner,

$$A_m^{(s)} = A_m^{(E)}, \qquad \Lambda^{(s)} = \tau_2^{-1/8}\Lambda^{(E)}, \qquad \bar\Lambda^{(s)} = \tau_2^{-1/8}\bar\Lambda^{(E)}, \qquad \varphi_{AB}^{(s)} = \tau_2^{-1/4}\varphi_{AB}^{(E)}. \quad (6.1)$$

In the following the superscript $^{(E)}$ will often be dropped. With these field normalizations a factor of τ_2 multiplies only the Maxwell term in the free action.

It will be convenient to absorb a power of e^{ϕ} into the field strengths so that (from (2.17)) the combination $\hat F^{\pm} = \tau_2^{1/2} F^{\pm}$ transforms with a phase under $SL(2,Z)$,

$$\hat F^{\pm} \to \left(\frac{c\bar\tau + d}{c\tau + d}\right)^{\pm 1/2} \hat F^{\pm}. \qquad (6.2)$$

The fermion fields also transform with a phase,

$$\bar\Lambda \to \left(\frac{c\bar\tau + d}{c\tau + d}\right)^{1/4} \bar\Lambda, \qquad \Lambda \to \left(\frac{c\bar\tau + d}{c\tau + d}\right)^{-1/4} \Lambda, \qquad (6.3)$$

while the scalar field φ_{AB} is inert.[7]

6.1. Instanton induced interactions

The tree-level open-string amplitude with external fermions is a well-known generalization of (2.6) that gives rise to higher derivative interactions of the form,

$$\alpha'^4 \int d^4x \det e^{(s)} \tau_2 (\partial^2 \Lambda^{(s)})^2 (\partial \bar\Lambda^{(s)})^2 = \alpha'^4 \int d^4x \det e\, \tau_2 (\partial^2\Lambda)^2 (\partial\bar\Lambda)^2, \qquad (6.4)$$

where the precise contractions between the indices are defined by the kinematic factor of the four-fermion amplitude [27]. Similarly, there are terms of the same dimension of the schematic form $\partial \hat F^{+} \partial \hat F^{-} \partial\bar\Lambda \partial^2 \Lambda$ and $(\partial^2\varphi)^2 \partial\bar\Lambda\partial^2\Lambda$, which are also modular invariant.

[7] These transformation rules can be obtained by considering $N=4$ Maxwell theory coupled to $N=4$ supergravity [41,42].

It seems plausible that the coupling constant dependence again enters via the modular function $h(\tau,\bar\tau)$ that is the coefficient of the $(\partial\hat F)^4$ term. This is supported by the discussion in section 5.3 where the instanton contributions to these terms were seen to be related by $\mathcal{N}=4$ supersymmetry transformations since they were obtained from a supersymmetric generating function.

Similarly, there are terms that transform with non-zero modular weights. Although the tree-level expressions for these interactions are complicated to analyze it is easy to deduce their presence from the instanton induced interactions. For example, there is an eight-fermion term which has a tree-level contribution,

$$\alpha'^8 \int d^4x \det e^{(s)}\, \tau_2 (\partial\bar\Lambda^{(s)})^8 = \alpha'^8 \int d^4x \det e\, \tau_2 (\partial\bar\Lambda)^8. \tag{6.5}$$

Since $\bar\Lambda^8$ transforms with a phase $(c\tau+d)^2/(c\bar\tau+d)^2$ the modular-invariant interaction must have the form,

$$\alpha'^8 \int d^4x \det e\, h^{(2,-2)}(\tau,\bar\tau)\, (\partial\bar\Lambda)^8, \tag{6.6}$$

where the modular form $h^{(2,-2)}(\tau,\bar\tau)$ transforms with a compensating phase $(c\bar\tau+d)^2/(c\tau+d)^2$ and has a large τ_2 (small coupling constant) expansion that starts with the tree-level term. In a similar manner there are interactions of the form

$$\alpha'^6 \int d^4x \det e\, h^{(1,-1)} (\partial\bar\Lambda)^4\, (\partial^2\varphi)^2, \qquad \alpha'^6 \int d^4x \det e\; h^{(1,-1)} (\partial\bar\Lambda)^4 \hat F^+ \partial^2 \hat F^-, \qquad \ldots. \tag{6.7}$$

This structure is similar to the structure of higher derivative terms in the bulk type IIB action which are of the same dimension as R^4 [43]. In that case there are interactions with integer weights $(w,-w)$ with $0\le w\le 12$ while in the present case the weights span the range $0\le w\le 2$. Following the same path as in that case, it is reasonable to conjecture that the relations between the modular forms $h^{(w,-w)}$ are obtained by applying successive modular covariant derivatives on the modular function,[8] $h^{(0,0)}(\tau,\bar\tau)\equiv h(\tau,\bar\tau)=\ln|\tau_2\eta(\tau)^4|$. Thus,

$$\begin{aligned} h^{(1,-1)} &= D_0 h^{(0,0)}(\tau,\bar\tau) = i\tau_2 \frac{\partial}{\partial\tau} h^{(0,0)} = -\frac{\pi}{6}\tau_2 \hat E_2 \\ &= -\frac{\pi}{6}\tau_2 + \frac{1}{2} + 4\pi\tau_2 \sum_{N=1}^{\infty}\sum_{m|N} m e^{2i\pi\tau N}, \end{aligned} \tag{6.8}$$

[8] The notation $h^{(0,0)}\equiv h$ is suited to the the generalization to modular forms of non-zero weight.

where E_2 is the second Eisenstein series (which is holomorphic but not modular covariant) while $\hat{E}_2$ is the non-holomorphic Eisenstein series of weight $(2,0)$.

$$\hat{E}_2 = E_2 - \frac{3}{\pi\tau_2}, \qquad E_2 = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}, \tag{6.9}$$

where $q = e^{2\pi i\tau}$. More generally, the covariant derivative acting on a modular form $h^{(w,-w)}$ is $D_w = i(\tau_2\partial/\partial\tau - iw/2)$ converts it to a form $h^{(w+1,-w-1)}$. Thus, applying a covariant derivative to $h^{(1,-1)}$ gives

$$\begin{aligned} h^{(2,-2)} &= D_1 h^{(1,-1)} = i\big(\tau_2\frac{\partial}{\partial\tau} - \frac{i}{2}\big)h^{(1,-1)} \\ &= -\frac{\pi}{6}\tau_2^2\big(i\frac{\partial}{\partial\tau} + \frac{1}{\tau_2}\big)\hat{E}_2 \\ &= -\frac{\pi}{36}\tau_2^2\big(E_4 - \hat{E}_2^2\big) \end{aligned} \tag{6.10}$$

where we have used the fact that

$$\big(i\frac{\partial}{\partial\tau} + \frac{1}{\tau_2}\big)\hat{E}_2 = \frac{\pi}{6}(E_4 - \hat{E}_2^2) \tag{6.11}$$

and the fourth Eisenstein series is defined by

$$E_4 = 1 + 240\sum\frac{n^3q^n}{1-q^n}. \tag{6.12}$$

The instanton expansion of $h^{(2,-2)}$ is given by

$$h^{(2,-2)} = -\frac{\pi}{6}\tau_2 + \frac{1}{4} + 4\pi\tau_2\sum_{N=1}^{\infty}\sum_{m|N} m e^{2\pi i\tau N} - 8\pi\tau_2^2\sum_{N=1}^{\infty}\sum_{m|N} mN e^{2\pi i\tau N}. \tag{6.13}$$

All the expressions $h^{(w,-w)}$ possess tree-level and one-loop terms that correspond to perturbative effects in the world-volume of the D3-brane. The instanton contributions to the expansions of $h^{(0,0)}$ in (1.2), $h^{(1,-1)}$ in (6.8) and $h^{(2,-2)}$ in (6.13) are multiplied by different powers of τ_2. The leading terms in the weak coupling limit ($\tau_2 = g^{-1} \to \infty$) have powers τ_2^0, τ_2 and τ_2^2, respectively. These powers correspond to the terms with $r = 4$, $r = 6$ and $r = 8$ in (5.23), which summarizes the results of the explicit leading-order D-instanton calculations, which automatically take account of supersymmetry constraints. We therefore have some evidence that the conjectured form $SL(2,Z)$-covariant prefactors $h^{(1,-1)}$ and $h^{(2,-2)}$ are given (up to overall constant normalizations) by (6.8) and (6.10), respectively.

In [5] the relation of the D3 brane to the M5 brane wrapped on a two-torus was used to show that $h^{(0,0)}$ can be obtained from a one loop calculation in the world-volume theory of the M5 brane. It would be interesting to use an analogous one-loop calculation to reproduce the conjectured form of $h^{(1,-1)}$ and $h^{(2,-2)}$.

7. Closed-string interactions

The effective action of a D3-brane depends in a crucial manner on the geometry that describes the embedding of the world-volume in the target space-time. This means that in addition to the dependence on the fields in the Yang–Mills supermultiplet it is important to understand terms in the effective action that involve the gravitational sector pulled back to the world-volume. These have been discussed to some extent in the literature. For example, terms of the general form R^2 arising from tree-level scattering of a graviton from a Dp-brane were obtained in [2,44,45] where the two-point tree-level graviton amplitude was written as

$$
\begin{aligned}
A^{tree}_{R^2}(h^{(1)}, p_1; h^{(2)}, p_2) &= -\frac{1}{8} T_{(p)}\ \alpha'^2\ K(1,2)\ \frac{\Gamma(-\alpha' t/4)\Gamma(\alpha' q^2)}{\Gamma(1-\alpha' t/4+\alpha' q^2)} \\
&= \frac{1}{2} T_{(p)}\ K(1,2)\ \left(\frac{1}{q^2 t} + \frac{\pi^2 \alpha'^2}{24} + o(\alpha'^4)\right).
\end{aligned}
\tag{7.1}
$$

Here $h^{(1)}_{\mu\nu}$ and $h^{(2)}_{\mu\nu}$ ($\mu = 0, \ldots, 9$) are the polarization vectors for the on-shell gravitons, $q^2 = p_1 \cdot D \cdot p_1/2$ is the square momentum flowing along the world-volume of the Dp-brane and $t = -2 p_1 \cdot p_2$ is the momentum transfer in the transverse directions and the matrix D is the diagonal matrix

$$
D^{\nu}_{\mu} = \begin{cases} \delta_{\mu}{}^{\nu} & (\mu, \nu = 0, \ldots, p), \\ -\delta^{\nu}_{\mu} & (\mu, \nu = p+1, \ldots, 9). \end{cases}
\tag{7.2}
$$

The kinematic factor K is expressed in terms of the tensor t_8 which appear in the four open-string scattering amplitude,

$$
K(1,2) = t_8^{\mu_1 \nu_1 \rho_1 \lambda_1 \mu_2 \nu_2 \rho_2 \lambda_2}\ D^{\bar\rho_1}_{\rho_1} D^{\bar\lambda_1}_{\lambda_1}\ D^{\bar\rho_2}_{\rho_2} D^{\bar\lambda_2}_{\lambda_2}\ h^{(1)}_{\mu_1 \bar\rho_1} k^1_{\nu_1} k^1_{\bar\lambda_1} h^{(2)}_{\mu_2 \bar\rho_2} k^2_{\nu_2} k^2_{\bar\lambda_2}.
\tag{7.3}
$$

Using the definition (7.2) and the momentum invariants q^2 and t the kinematic factor (7.3) can be written as

$$
K(1,2) = \left(2q^2\, a_1 + \frac{t}{2}\, a_2\right),
\tag{7.4}
$$

with

$$
\begin{aligned}
a_1 &= \mathrm{tr}(h^{(1)} \cdot D)\, p_1 \cdot h^{(2)} \cdot p_1 - p_1 \cdot h^{(2)} \cdot D \cdot h^{(1)} \cdot p_2 - p_1 \cdot h^{(2)} \cdot h^{(1)} \cdot D \cdot p_1 \\
&\quad p_1 \cdot h^{(2)} \cdot h^{(1)} \cdot D \cdot p_1 - p_1 \cdot h^{(2)} \cdot h^{(1)} \cdot p_2 + q^2\, \mathrm{tr}(h^{(1)} \cdot h^{(2)}) + \Big\{1 \leftrightarrow 2\Big\} \\
a_2 &= \mathrm{tr}(h^{(1)} \cdot D)\, (p_1 \cdot h^{(2)} \cdot D \cdot p_2 + p_2 \cdot D \cdot h^{(2)} \cdot p_1 + p_2 \cdot D \cdot h^{(2)} \cdot D \cdot p_2) \\
&\quad + p_1 \cdot D \cdot h^{(1)} \cdot D \cdot h^{(2)} \cdot D \cdot p_2 - p_2 \cdot D \cdot h^{(2)} \cdot h^{(1)} \cdot D \cdot p_1 + q^2\, \mathrm{tr}(h^{(1)} \cdot D \cdot h^{(2)} \cdot D) \\
&\quad - q^2\, \mathrm{tr}(h^{(1)} \cdot h^{(2)}) - \mathrm{tr}(h^{(1)} \cdot D) \mathrm{tr}(h^{(2)} \cdot D)\, (q^2 - t/4) + \Big\{1 \leftrightarrow 2\Big\}.
\end{aligned}
\tag{7.5}
$$

The form of this amplitude, together with some mild topological input, leads to an explicit expression for the tree-level contribution to the CP even (curvature)^2 terms in the action [5],

$$\mathcal{L}^{(p)}_{CP-even} = \frac{\tau_2}{3\times 2^7\pi}\Big((R_T)_{\alpha\beta\gamma\delta}(R_T)^{\alpha\beta\gamma\delta} - \\ - 2(R_T)_{\alpha\beta}(R_T)^{\alpha\beta} - (R_N)_{\alpha\beta ab}(R_N)^{\alpha\beta ab} + 2\bar{R}_{ab}\bar{R}^{ab}\Big). \tag{7.6}$$

The notation in this expression is reviewed in appendix B. As in the case of the open-string interactions discussed earlier the expressions in brackets are modular invariant and a $SL(2,Z)$-invariant action is obtained when the factor of τ_2 is replaced by a modular function. The CP-odd curvature terms in the Dp-brane action are determined by an anomaly-cancelling argument.

General arguments were given in [5] that the modular function describing the exact scalar field dependence of the non-perturbative contributions to the (curvature)^2 terms involving the tangent bundle (including the terms in the CP-odd part of the action) is the function $h^{(0,0)}$ defined by (1.1). However, little was said concerning non-perturbative normal bundle effects which is our main interest in this section. The dependence of the tangential and longitudinal pull-backs of the curvature (R_T and R_N) as well as $\bar{R}$ on the second fundamental form leads to a dependence on the scalar fields φ_{AB}. Thus, the terms $\Omega^a_{mp}\Omega^b_{nq} - \Omega^a_{mq}\Omega^b_{np}$ in R_T and $\Omega^a_{mp}\Omega^b_{nq} - \Omega^b_{mp}\Omega^a_{nq}$ in R_N together with the terms bilinear in Ω that enter into $\bar{R}$ give terms in (7.6) that are of the form $(\partial^2\varphi)^4$ and indeed correspond to the open-string scalar field interactions that we discussed in the earlier sections [9].

We are now in a position to say more by generalizing the arguments of the preceding sections to the situation in which there are closed-string fields that couple via closed-string vertex operators. There are three kinds of contributions corresponding to the three possible boundary conditions on a disk: (a) The disk with D3-brane conditions gives the standard tree-level gravitational contributions to the Born–Infeld action; (b) The disk with D-instanton boundary conditions determines the zero modes of the gravitational fields in the D-instanton background and will give rise to (curvature)^2 terms in the effective action; (c) The gravitational coupling to a disk with N and D conditions (and two twist operators) – these effects are of relevance to the situation in which there is a non-zero background B_{mn} field which will be commented on in the conclusion. We will here give a detailed discussion of case (b) only.

[9] We are grateful to P. Bain for a useful correspondence on this point.

7.1. (Curvature)^2 and related terms

In the absence of a D3-brane the one-point function of any field in the IIB supergravity multiplet in a D-instanton background may be obtained by coupling the field to a disk with Dirichlet boundary conditions in all directions [6]. This multiplet of one-point functions is generated by applying the sixteen broken supersymmetries, $\theta^{\mathcal{A}} Q_{\mathcal{A}}$, to the dilaton one-point function, where $\theta^{\mathcal{A}}$ ($\mathcal{A} = 1, \ldots, 16$) is a Weyl Grassmann spinor. In the present context this is equivalent to acting on the dilaton one-point function with the broken supersymmetries which are generated by $\eta^A Q_A$ and $\rho_A \bar{Q}^A$. The parameters η and ρ are identified with the components of θ. As described more fully in [6], these one-point functions are contained in the type IIB scalar on-shell superfield of [46]. This is a field $\tilde{\Phi}(x, \theta, \bar{\theta})$ that satisfies the chirality condition $\bar{D}\tilde{\Phi} = 0$ as well as the 'on-shell' condition $D^4\tilde{\Phi} = \bar{D}^4\bar{\tilde{\Phi}}$ (where D is the type IIB covariant derivative) that restrict its components to satisfy the equations of motion. This field has the expansion

$$\tilde{\Phi}(\theta) = a + 2\theta\gamma^0\hat{\Lambda} + \frac{1}{24}\theta\gamma^0\gamma^{\mu\nu\rho}\theta\, G_{\mu\nu\rho} + \ldots - \frac{i}{48}\theta\gamma^0\gamma^{\rho\mu\nu}\theta\,\theta\gamma^0\gamma_\rho{}^{\lambda\tau}\theta\, R_{\mu\lambda\nu\tau} + \ldots, \tag{7.7}$$

which terminates after the eighth power of θ. In this expression γ^μ ($\mu = 0, 1, \ldots, 9$) are the ten-dimensional (euclidean) gamma matrices and the complex fluctuation of the scalar field is denoted by a, which is defined by

$$\begin{aligned} \tau &= \chi + \frac{i}{g} + \frac{1}{g}(\hat{C}^{(0)} - i\hat{\phi}) \\ &= \tau_0 + \frac{1}{g}a, \end{aligned} \tag{7.8}$$

(where the constant background scalar field is now denoted by $\tau_0 = \chi + ig^{-1}$ and the hats denote fluctuations of fields) and transforms with $SL(2,Z)$ weight $(2,-2)$. The complex dilatino, [10] $\hat{\Lambda}$, transforms with weight $(3/2, -3/2)$, while $G_{\mu\nu\rho}$ is the field strength of a complex linear combination of $NS \otimes NS$ and $R \otimes R$ antisymmetric tensor potentials that transforms with weight $(1, -1)$. The complex gravitino of weight $(3/2, -3/2)$ occurs as the coefficient of the term of order θ^3 while the $SL(2,Z)$-invariant scalar Riemann curvature occurs as the coefficient of the θ^4 term (together with ∂F_5, where F_5 is the self-dual $R \otimes R$ field strength). The higher powers of θ have coefficients that are derivatives acting on the complex conjugate fields.

We here want to include a D3-brane in the $0, 1, 2, 3$ directions. The sixteen supersymmetries broken by the D-instanton are now distinguished by the fact that the ρ

[10] The dilatino is here represented by the symbol $\hat{\Lambda}$ to avoid confusion with the gaugino of the open-string sector.

supersymmetries are also broken by the D3-brane while the η supersymmetries are unbroken by the D3-brane. More precisely, the extra condition on the ten-dimensional chiral spinor θ that corresponds to the supersymmetries that are preserved on the D3-brane is,

$$\mathcal{P}_+ \, \theta = 0, \tag{7.9}$$

where $\mathcal{P}_\pm = (1 \pm \gamma^{0123})/2$ is a projection operator. The solution of this equation has eight independent components that are associated with the η supersymmetries that act as shifts on M'. Likewise, the remaining eight components satisfy $\mathcal{P}_- \theta = 0$. These are to be identified with the with the ρ supersymmetries that act as shifts on $\lambda^{\dot\alpha}_A$ and $\bar\Lambda^{\dot\alpha}_A$. The zero modes resulting from the breaking of the bulk supersymmetries are therefore obtained by expressing the superfield $\tilde\Phi$ as a power series in M' and λ.

For example, one contribution to the coupling of the graviton to the disk is obtained by acting four times with the broken η supersymmetries on the dilaton one-point function, which picks out the fourth power of θ in the superfield $\tilde\Phi$,

$$\langle h \rangle_4 = h_{\mu\nu} k_\lambda k_\rho \theta \gamma^0 \gamma_\tau{}^{\mu\lambda} \theta \, \theta \gamma^0 \gamma^{\tau\nu\rho} \theta. \tag{7.10}$$

In the 'minimal' case that we considered earlier, in which all the μ and λ fermions are integrated out by bringing down factors of $\lambda\mu\bar{w}$ and $\lambda\bar{\mu}w$ from $e^{-S_{-1}}$, the non-zero graviton D-instanton induced two-point function is given purely by the integration over the eight components of θ that correspond to the M' coordinates,

$$\begin{aligned} A^{inst}_{R^2} &= \mathcal{C} \int \frac{d^3 W^c}{W_0} d^6\chi d^8 M' \, \langle h_1 \rangle_4 \, \langle h_2 \rangle_4 \\ &= \mathcal{C} \int \frac{d^3 W^c}{W_0} d^6\chi d^8 M' \, e^{-S_{-1}} h^{(1)}_{\mu_1\nu_1} k^1_{\lambda_1} k^1_{\rho_1} \theta \gamma^0 \gamma^{\tau_1\mu_1\lambda_1} \theta \, \theta \gamma^0 \gamma^{\tau_1\nu_1\rho_1} \theta \\ &\quad h^{(2)}_{\mu_2\nu_2} k^2_{\lambda_2} k^2_{\rho_2} \theta \gamma^0 \gamma^{\tau_2\mu_2\lambda_2} \theta \, \theta \gamma^0 \gamma^{\tau_2\nu_2\rho_2} \theta. \end{aligned} \tag{7.11}$$

In order to evaluate the fermionic integrals and show that this expression is proportional to the kinematic factor (7.4) we need to project θ onto its λ and M' components. It is convenient to use a special frame in which $SO(10)$ is broken to $SO(2) \times SO(8)$, where the $SO(2)$ refers to rotations in the $\mu = 0, 9$ plane and the momenta and polarizations of the graviton wave functions are chosen to be independent of the $\mu = 0, 9$ directions. The D(-1)/D3 background breaks the $SO(8)$ to $SO(5) \times SO(3)$ (the two factors being associated with the directions $4, 5, 6, 7, 8$ and $1, 2, 3$, respectively), which is also a subgroup of the full $SO(6) \times SO(4)$ symmetry group. The chiral $SO(6) \times SO(4)$ bispinor M' can then be identified with a chiral $SO(8)$ spinor. More explicitly, the sixteen-component spinor θ decomposes under $SO(8)$ into two spinors of opposite chiralities, $\theta = M_c + M_s$, where $M_c = (1 + i\gamma^{09})\theta/2$ and $M_s = (1 - i\gamma^{09})\theta/2$. Since θ satisfies the condition (7.9) it is possible to write $\theta = 2\mathcal{P}_- \, \eta_s$ where η_s is a $\mathbf{8_s}$ spinor. It follows that $M_s = \eta_s$ and

$M_c = \hat\gamma^{123}\,\eta_s$. It is then straightforward to show (using an explicit representation of $SO(10)$ gamma matrices in terms of $SO(8)$ gamma matrices) that

$$\begin{aligned} h_{\mu\nu}k_\rho k_\lambda\, \theta\gamma^0\gamma^{\tau\mu\lambda}\theta\, \theta\gamma^0\gamma^{\tau\nu\rho}\theta &\to h_{\mu\nu}k_\lambda k_\rho\ M_c\hat\gamma^{\mu\lambda}M_c\ M_s\hat\gamma^{\nu\rho}M_s \\ &= h_{\mu\nu}k_\lambda k_\rho\ \eta_s\hat\gamma^{\mu\lambda}\eta_s\ \eta_s\hat\gamma^{123}\hat\gamma^{\nu\rho}\hat\gamma^{123}\eta_s \\ &= h_{\mu\bar\nu}k_\lambda k_{\bar\rho}\ D^{\bar\rho}_\rho D^{\bar\nu}_\nu\ \eta_s\hat\gamma^{\mu\lambda}\eta_s\ \eta_s\hat\gamma^{\nu\rho}\eta_s. \end{aligned} \tag{7.12}$$

On the right hand side of this equation $\hat\gamma$ indicates a 8×8 $SO(8)$ gamma matrix, and M_s, M_c and η_s are now eight-component $SO(8)$ spinors (and the vector indices are in the range $\mu,\nu,\lambda,\rho = 1,\cdots,8$) and use has been made of $SO(8)$ Fierz relations.

Inserting (7.12) into (7.11) and identifying η_s with M' leads once again to the standard integration over the product of components of a fermionic $SO(8)$ spinor (as in (5.13)) that results in contractions with the tensor, t_8. The resulting D-instanton induced two-graviton amplitude is

$$A^{inst}_{R^2} = \mathcal{C}\ (\alpha')^2\ e^{2\pi i\tau}\ t_8^{\mu_1\nu_1\rho_1\lambda_1\mu_2\nu_2\rho_2\lambda_2}\ D^{\bar\rho_1}_{\rho_1}D^{\bar\lambda_1}_{\lambda_1}\ D^{\bar\rho_2}_{\rho_2}D^{\bar\lambda_2}_{\lambda_2}\ h^{(1)}_{\mu_1\bar\rho_1}k^1_{\nu_1}k^1_{\bar\lambda_1}\, h^{(2)}_{\mu_2\bar\rho_2}k^2_{\nu_2}k^2_{\bar\lambda_2} + c.c. \tag{7.13}$$

(absorbing a combination of constants into the definition of $\mathcal{C}$). After covariantizing this expression it is proportional to the $(\alpha')^2$ term of the tree level result (7.3).

A similar discussion leads to the other instanton-induced interactions of bulk supergravity fields together with the D3-brane supersymmetric Maxwell fields that break 24 of the original 32 supersymmetries (i.e., are in the 'minimal' sector). More generally, the presence of closed string sources leads to sectors in which more supersymmetry is broken. As we saw, this arises from terms in which some, or all, of the factors of μ and $\bar\mu$ are soaked up by $\mu\bar\mu$ bilinears in the expansion of $e^{S_{-1}}$. This is the situation in which some, or all, powers of λ are taken from the closed-string superfield $\tilde\Phi(M',\lambda)$. The sector in which all the λ's are soaked up by the closed-string sources is the one in which all 32 supersymmetries are broken. In this case the D-instanton carries the same set of zero modes as the isolated bulk D-instanton and behaves independently of the D3-brane.

8. Discussion

We have discussed some low energy aspects of terms of order ${\alpha'}^4$ in the low energy expansion of the world-volume action of a D3-brane. The effects of a D-instanton on such higher-derivative interactions were obtained by explicit integration over the collective coordinates. This resulted in a variety of interactions between the world-volume fields with a kinematic structure that matches the tree-level and one-loop interactions that arise in the low energy limit of perturbative open string theory. This motivated conjectures for the exact non-perturbative $SL(2,Z)$-invariant interactions. Furthermore, these combine

with terms quadratic in the tangent and normal curvatures induced on the world-volume where the dependence on the second fundamental form is built in by interactions of the open-string scalar field, φ^c.

The lowest order contributions to these instanton amplitudes are associated with disk diagrams with insertions of an even number of twist operators (which are the vertex operators for the massless ND modes) that change the boundary condition from Neumann to Dirichlet. The integration over the bosonic moduli W^c and χ^a is gaussian and peaks at the origin where the D-instanton coincides with the D3-brane. Such effects of D-instantons inside D-branes have appeared in other contexts, for example there are instanton corrections to $\mathrm{tr}(F^4), (tr(F^2))^2$ and related terms in the world-volume of D7-branes. Such terms can be calculated in special situations [47,48] using heterotic/type I duality and the modular functions appear in that case are again related to (1.1).

In our discussion we have not taken the limit of [35], in which gravity decouples from the excitations on the brane and which is at the heart of the AdS/CFT correspondence [49]. The effects we have found are therefore different from the instanton effects found in [18,50] which relate the induced D-instanton interactions to correlators in the CFT. In the AdS/CFT decoupling limit (the near-horizon limit of the classical D3-brane geometry) the D-instanton exists on the Higgs branch of the moduli space and the 'center of mass' U(1) gauge field discussed in this paper decouples.

In the gauge field theory description of $Dp/Dp+4$ systems the singularities associated with abelian instantons or 'small' non-abelian instantons are often problematic. The non-zero length scale that appears in string theory in a trivial background resolves these singularities which reappear in the low energy decoupling limit that reduces to the field theory in the absence of background bulk fields. However, it is well-known that such small instanton singularities are also smoothed out by the presence of a background antisymmetric tensor field, $B_{\mu\nu}$ [19,21]. In this case the decoupling limit is non-trivial even in the abelian case and is known to correspond to a non-commutative version of the gauge theory [21,40]. The background field is introduced by coupling the B_{mn} closed-string vertex operator to the disk with two twist operators on the boundary. For constant B this is a total derivative and the result is a term in the action proportional to $\langle B\rangle = B_{mn}W^{mn}$ which combines with the F_{mn} term (4.1) of section 4 to give the dependence on $2\pi\alpha'\mathcal{F} = 2\pi\alpha' F + B$ that is required in order to maintain the antisymmetric tensor gauge invariance. The action of the eight broken supersymmetries on $\langle B\rangle$ generates a supermultiplet of one-point functions that combine with the open-string supermultiplet Φ_{mn} of section 4. This description is valid for weak B_{mn}. A more complete analysis of this situation would take into account the fact that the background field induces a shift in the Neumann boundary conditions on the disk (turning them into Dirichlet conditions in the large B limit).

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Appendix A. Conventions

We are interested in properties of $SO(4)$ and $SO(6)$ spinors. For the $SO(4) = SU(2)_L \otimes SU(2)_R$ case we will use the conventions of Wess and Bagger [51] with the choice of representation of 4×4 gamma matrices,

$$\gamma^n = \begin{pmatrix} 0 & \sigma^n \\ \bar{\sigma}^n & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.1}$$

where the four matrices $\sigma^n_{\alpha\dot{\beta}}$'s are $\sigma^0 = I$ $(n = 0)$ and $i\sigma^i$ $(n = 1, 2, 3)$ where σ^i are the Pauli matrices (and the factor of i comes from the continuation to euclidean signature). A bar denotes the reversed assignment of spinor chiralities so that the indices on $\bar{\sigma}^n$ are $\bar{\sigma}^n_{\dot{\alpha}\beta}$. Spinor indices are raised by means of the antisymmetric tensors, $\epsilon^{\alpha\beta}$ and $\epsilon^{\dot{\alpha},\dot{\beta}}$. The $SO(4)$ group generators can be written as

$$\gamma^{mn} = \frac{1}{4}\left[\gamma^m, \gamma^n\right] = i\begin{pmatrix} \sigma^{mn} & 0 \\ 0 & \bar{\sigma}^{mn} \end{pmatrix}, \tag{A.2}$$

where,

$$\sigma^{mn}_{\alpha\beta} = \frac{1}{2}\left(\sigma^m\bar{\sigma}^n - \sigma^n\bar{\sigma}^m\right)_{\alpha\beta}, \qquad \bar{\sigma}^{mn}_{\dot{\alpha}\dot{\beta}} = \frac{1}{2}\left(\bar{\sigma}^m\sigma^n - \bar{\sigma}^n\sigma^m\right)_{\dot{\alpha}\dot{\beta}}. \tag{A.3}$$

These are the generators of Lorentz transformations on chiral spinors satisfying,

$$\epsilon^{mnpq}\sigma_{pq} = -2\sigma^{mn}, \qquad \epsilon^{mnpq}\bar{\sigma}_{pq} = 2\bar{\sigma}^{mn} \tag{A.4}$$

so they project on self-dual and anti self-dual tensors respectively. With this definition the coupling of $F^{\pm}$ is given by $\sigma^{mn}F^-_{mn}$ and $\bar{\sigma}^{mn}F^+_{mn}$.

The 'tHooft symbol η^c_{mn} maps the self-dual $SO(4) = SU(2)_L \times SU(2)_R$ tensor into the adjoint of one $SU(2)$ subgroup, while $\bar{\eta}^c_{mn}$ performs the mapping on the conjugate representations, so that,

$$\sigma^{mn}_{\alpha\beta} = \eta^c_{mn}\,\tau^c_{\alpha\beta}, \qquad \bar{\sigma}^{mn}_{\dot{\alpha}\dot{\beta}} = \bar{\eta}^c_{mn}\,\bar{\tau}^c_{\dot{\alpha}\dot{\beta}}. \tag{A.5}$$

These symbols may be explicitly represented in the form,

$$\begin{aligned} \eta^c_{mn} &= \bar{\eta}^c_{mn} = \epsilon_{cmn}, \qquad m, n \in \{1, 2, 3\}, \\ \bar{\eta}^c_{4n} &= -\eta^c_{4n} = \delta_{cn}, \\ \eta^c_{mn} &= -\eta^c_{nm}, \qquad \bar{\eta}^c_{mn} = -\bar{\eta}^c_{nm}. \end{aligned} \tag{A.6}$$

We also note the formula for the contraction of two η symbols,

$$\delta^{c_1c_2}\eta^{c_1}_{m_1n_1}\eta^{c_2}_{m_2n_2} = \delta^{m_1m_2}\delta^{n_1n_2} - \delta^{m_1n_2}\delta^{m_2n_1} + \epsilon^{m_1n_1m_2n_2}. \tag{A.7}$$

The $SO(6) = SU(4)$ gamma matrices are 8×8 matrices that may be represented by,

$$\Gamma^a = \begin{pmatrix} 0 & \Sigma^a \\ \bar{\Sigma}^a & 0 \end{pmatrix}, \qquad \Gamma_7 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{A.8}$$

where the matrices Σ^a_{AB} and $\bar{\Sigma}^{a\ AB}$ are the Clebsch–Gordan coefficients that couple two $\mathbf{4}$'s of $SU(4)$ to a $\mathbf{6}$ and two $\bar{\mathbf{4}}$'s to a $\mathbf{6}$, respectively. These can also be written in terms of the 'tHooft symbols,

$$\begin{aligned} \Sigma^a_{AB} &= \eta^c_{AB}, \qquad (a = 1,2,3); \qquad \Sigma^a_{AB} = i\bar{\eta}^c_{AB}, \qquad (a = 4,5,6) \\ \bar{\Sigma}^{AB}_a &= -\eta^c_{AB}, \qquad (a = 1,2,3); \qquad \bar{\Sigma}^{AB}_a = i\bar{\eta}^c_{AB}, \qquad (a = 4,5,6). \end{aligned} \tag{A.9}$$

The $SO(6)$ generators can be represented by

$$\Gamma^{ab} = \frac{i}{4}\left[\Gamma^a, \Gamma^b\right] = i \begin{pmatrix} \Sigma^{ab} & 0 \\ 0 & \bar{\Sigma}^{ab} \end{pmatrix}. \tag{A.10}$$

The charge conjugation operator for the $SO(4)$ group is

$$C_4 = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}, \tag{A.11}$$

while the charge conjugation operator for $SO(6)$ is

$$C_6 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}. \tag{A.12}$$

Appendix B. Curvature induced from embeddings

This appendix gives a very brief summary of some standard notation concerning embedded $(p+1)$-dimensional sub-manifolds (for further details see [52,53] and the summary in [5]). The embedding coordinates $Y^\mu(x^m)$ (where $\mu = 0, \cdots, 9$ is the target space-time index and $m = 0, \cdots p$ the world-volume index) describe the position of the $(p+1)$-dimensional world-volume in the target space. The quantity $\partial_m Y^\mu$ defines a local frame for the tangent bundle while a frame for the the normal bundle, ξ^μ_a $(a = p+1, \cdots 9)$, is defined by

$$\xi^\mu_a\ \xi^\nu_b\ G_{\mu\nu} = \delta_{ab} \quad \text{and} \quad \xi^\mu_a\ \partial_m Y^\nu\ G_{\mu\nu} = 0\ . \tag{B.1}$$

Both $\partial_m Y^\mu$ and ξ^μ_a transform as vectors under target-space reparameterization. The former are vectors of world-volume reparametrizations, while the latter transform as vectors under local $SO(9-p)$ rotations of the normal bundle. Correspondingly, the metric on the D-brane world-volume can be decomposed as

$$G^{\mu\nu} = \partial_m Y^\mu \partial_n Y^\nu\ g^{mn} + \xi^\mu_a \xi^\nu_b\ \delta^{ab}\ . \tag{B.2}$$

For many purposes it is convenient to use the static gauge in which $Y^m(x^m) = x^m$ (when $\mu = 0, 1, \ldots, p-1$). In this gauge the transverse coordinates are Y^a (where $\mu = p, \ldots, 9$) and are interpreted as the scalar fields $\varphi^a(x^m)$, that enter into the Dp-brane action.

Any tensor can be pulled back from the target space onto the tangent or normal bundles by contraction with the local frames. Thus, the pull-back of the bulk (target-space) metric is the induced world-volume metric,

$$g_{mn} = G_{\mu\nu}\, \partial_m Y^\mu \partial_n Y^\nu \ . \tag{B.3}$$

The Riemann tensor can be pulled back in different ways, depending on whether any of its four indices are contracted with the tangent or normal frame.

The target-space connection $\Gamma^\mu_{\nu\rho}$ can be constructed from the target-space metric while the world-volume connection $\Gamma_{T}{}^{m}_{n\gamma}$ can be constructed in terms of the induced metric. The connection on the normal bundle is a composite SO(9-p) gauge field

$$\omega_m^{ab} = \xi^{\mu,[a}\ \left(G_{\mu\nu}\partial_m + G_{\mu\sigma}\Gamma^\sigma_{\nu\rho}\partial_m Y^\rho\right)\ \xi^{\nu,b]}\ , \tag{B.4}$$

which may be defined by requiring that the normal frame be covariantly constant. The covariant derivative of the tangent frame, which is the second fundamental form, is given by

$$\Omega^\mu_{mn} = \Omega^\mu_{nm} = \partial_m\partial_n Y^\mu - (\Gamma_T)^\gamma_{mn}\ \partial_\gamma Y^\mu + \Gamma^\mu_{\nu\rho}\ \partial_m Y^\nu \partial_l Y^\rho\ , \tag{B.5}$$

and is a symmetric world-volume tensor and space-time vector. The tangent-space pull-back of Ω vanishes so that $\Omega^l_{mn} = 0$, which implies that the non-zero components are

$$\Omega^a_{mn} \equiv \Omega^\mu_{mn}\ \xi^{\nu,a} G_{\mu\nu}\ . \tag{B.6}$$

Totally geodesic embeddings are characterized by a vanishing second fundamental form.

The Gauss-Codazzi equations express the world-volume curvature R_T, constructed out of the affine connection Γ_T, and the field strength, R_N, of the $SO(9-p)$ gauge connection ω, to pull-backs of the space-time Riemann tensor together with combinations of the second fundamental form. These relations are

$$(R_T)_{mnpq} = R_{mnpq} + \delta_{ab}\left(\Omega^a_{mp}\Omega^b_{nq} - \Omega^a_{mq}\Omega^b_{np}\right) \tag{B.7}$$

and

$$(R_N)_{mn}{}^{ab} = -R_{mn}{}^{ab} + g^{pq}\left(\Omega^a_{mp}\Omega^b_{nq} - \Omega^b_{mp}\Omega^a_{nq}\right)\ . \tag{B.8}$$

Only if the Dp-brane world-volume is a totally geodesic manifold (so that $\Omega = 0$) do the curvature forms R_T and R_N coincide with the pull-backs of the bulk curvature. For embeddings in flat space-time these world-volume curvatures can be expressed entirely in terms of the second fundamental form Ω. From (B.5) we see that the presence of the terms bilinear in Ω in (B.7) and (B.8) translates into terms of quadratic and higher order in derivatives of the scalar fields.

The bulk Ricci tensor vanishes in the backgrounds of relevance to us. However, the tensor $\hat{R}_{ab} = g^{mn} R_{manb}$ is nonvanishing and enters in the expression (7.6) where the combination $\bar{R}_{ab}$ is defined by

$$\bar{R}_{ab} \equiv \hat{R}_{ab} + g^{mm'} g^{nn'}\Omega_{a|mn}\Omega_{b|m'n'}. \tag{B.9}$$

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